Asymmetrizing infinite trees

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Abstract

A graph G is asymmetrizable if it has a set of vertices whose setwise stablizer only consists of the identity automorphism. The motion m of a graph is the minimum number of vertices moved by any non-identity automorphism. It is known that infinite trees T with motion $m = \aleph_0$ are asymmetrizable if the vertex-degrees are bounded by 2^m. We show that this also holds for arbitrary, infinite m, and that the number of inequivalent asymmetrizing sets is $2^{|T|}$.

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1 Introduction

Given a graph G, a set $S \subseteq V(G)$ is called *asymmetrizing* if the identity is only element of $\operatorname{Aut} G$ which setwise fixes S. A graph is called *asymmetrizable*, if it has an asymmetrizing set.

Motivated by the use of asymmetric graphs in the construction of graphs with given automorphism group, asymmetrization was perhaps first studied in a 1977 paper by Babai [2], where he proved that every tree T in which all vertices have the same (finite or infinite) degree is asymmetrizable. This result was later reproved and generalised by Polat and Sabidussi [7, 9].

Much recent work on asymmetrization was motivated by a popular paper by Albertson and Collins [1], and a lot of it has focussed on the connection between asymmetrization and the concept of *motion* of a graph, which is defined as the minimum number of vertices moved by any non-identity automorphism. This connection was already noted by

Cameron, Neumann and Saxl in [4], where they studied asymmetrizing sets for permutation groups. For graphs the most notable recent result linking motion to asymmetrization is due to Babai [3]: he proved that each connected, locally finite graph with infinite motion is asymmetrizable thereby verifying the Infinite Motion Conjecture of Tucker [12].

The primary motivation for this paper is [6, Question 4], which asks whether each tree of motion $m > \aleph_0$ is asymmetrizable if its degrees are bounded by 2^m. Our main theorem, Theorem 1, answers it affirmatively.

Theorem 1. *Let m be an infinite cardinal and T be a tree whose degrees are bounded by 2^m. If the minimum number of vertices moved by each non-trivial automorphism of T is m, then T is asymmetrizable and the number of inequivalent asymmetrizing sets is $2^{|T|}$.*

The bound 2^m is sharp, because the tree consisting of more than 2^m rooted isomorphic asymmetric trees of order m whose roots are connected to a common vertex is not asymmetrizable. Note that Theorem 1 generalizes Babai's result from [2], because trees where all vertices have the same infinite degree α have motion α and thus satisfy the assumptions of the theorem.

In Section 4 we apply Theorem 1 to tree like-graphs. Regarding other applications we wish to point out that the methods of this paper can be used to derive results that are analogous to Theorem 1 for edge colorings.

Finally, it is worth mentioning that our methods only rely on ZFC and do not assume the Generalized Continuum Hypothesis, just as the papers [7, 9], whose results we use.

2 Preliminaries

A graph G is *asymmetrizable* if it has an *asymmetrizing* set of vertices, that is, a set $S \subseteq V(G)$ which is preserved only by the identity automorphism. If S is such a set, then its complement $V(G) \setminus S$ is also asymmetrizing. The definition allows that S or $V(G) \setminus S$ are empty.

Two asymmetrizing sets S of G and S' of G' are called *equivalent* if there exists an isomorphism φ from G to G' such that $\varphi(S_1) = S_2$. Following [9] we define the *asymmetrizing number* of G, denoted by $a(T)$, as the number of pairwise inequivalent asymmetrizing sets. Observe that $a(G) \leq 2^{|V(G)|}$ for all graphs, and that $a(G) = 2^{|V(G)|}$ for asymmetric graphs.

Let (T, w) denote the tree with root $w \in V(T)$, and denote by $\mathrm{Aut}(T, w)$ the subgroup of $\mathrm{Aut}(T)$ that fixes w. By slight abuse of notation, we call a subset $S \subseteq V(T)$ *asymmetrizing for* (T, w), if the identity is the only element of $\mathrm{Aut}(T, w)$ which fixes S setwise. Two asymmetrizing sets S and S' are called equivalent with respect to $\mathrm{Aut}(T, w)$, if there is an element of $\mathrm{Aut}(T, w)$ which maps S to S'. We define $a(T, w)$ as the number of inequivalent asymmetrizing sets of (T, w).

Let (T, w) be a rooted tree. For vertices x, y of (T, w) we let $x \geq y$ denote the fact that y lies on the unique path from the root to x. As usual, we say that y is the parent of x if $y < x$ and xy is an edge. We say x and x' are siblings if they have the same parent, and that x and x' are twins if they are siblings and if there is an automorphism which moves x to x' and fixes their parent. We call the set of twins of x the *similarity class* of x, denote it by $\bar{x}$, and set $\tau(x) = |\bar{x}|$. We always have $\tau(x) \geq 1$ since x is a twin of itself.

For a vertex x of (T, w) with parent y, we let T^x denote the component of $T - xy$ which contains x. We consider T^x as a rooted tree with root x and write $a(T, w; x)$ for the number of inequivalent asymmetrizing sets of T^x. If the rooted tree (T, w) is clear from the context, we write $a(x)$ instead of $a(T, w; x)$; in particular, in this case we also write $a(w)$ instead of $a(T, w)$. Let y be the parent of x and x'. Then clearly x and x' are twins if and only if T^x and $T^{x'}$ are isomorphic.

Let R_y be a set of representatives for the similarity classes of siblings of y. Then by [9, Theorem 2.3]

$$a(y) = \prod_{x \in R_w} \binom{a(x)}{\tau(x)}, \tag{1}$$

where $\binom{a}{\tau}$ denotes the usual binomial coefficient for finite a and τ. If a is infinite, then $\binom{a}{\tau}$ is a^τ if $\tau \leq a$, and 0 if $\tau > a$.

Equation (1) implies a helpful lemma that uses the concept of motion. Recall from the introduction that the motion $m(G)$ of a graph G is the least number of vertices moved by a non-identity automorphism of G. For asymmetric graphs the motion is not defined, but we use the convention that $m(G) > \alpha$ for any asymmetric graph and any cardinal α. In particular, a graph with fewer than α vertices has (by definition) motion $m(G) \geq \alpha$ if and only if it is asymmetric. This means that the order of a graph G with motion m is at least m unless G is asymmetric.

Lemma 2. *Let (T, w) be a rooted tree with motion $m \geq \aleph_0$, all of whose degrees are bounded by 2^m, and let $y \in V(T)$. If $a(x) = 2^{|T^x|}$ for all children x of y, then $a(y) = 2^{|T^y|}$.*

Proof. Let (T, w) and y satisfy the assumptions of the lemma. Then for all siblings x of y, the subtree T^x has motion m and is asymmetrizable.

Let x be a child of y such that $|T^x| < m$. Then T^x is asymmetric, and hence $a(x) = 2^{|T^x|}$. Moreover $\tau(x) = 1$, because otherwise the motion would be less than m.

Now let x be a child of y such that $|T^x| \geq m$. Then $a(x) \geq 2^m \geq \tau(x)$ and hence

$$\binom{a(x)}{\tau(x)} = (2^{|T^x|})^{\tau(x)} = 2^{|T^x|\,\tau(x)}.$$

Because $|T^x|\,\tau(x)$ is the total size of the union of the $V(T^z)$ for $z \in \bar{x}$, we conclude that

$$\binom{a(x)}{\tau(x)} = 2^{|\bigcup_{z \in \bar{x}} V(T^z)|}.$$

Substituting into Equation (1) we obtain

$$\begin{aligned} a(y) &= \prod_{x\in R_y} \binom{a(x)}{\tau(x)} = \prod_{x\in R_w} 2^{|\bigcup_{z\in\bar{x}} V(T^z)|} \\ &= 2^{\sum_{x\in R_w} |\bigcup_{z\in\bar{x}} V(T^z)|} = 2^{|\bigcup_{x\in R_w} \bigcup_{z\in\bar{x}} V(T^z)|} \\ &= 2^{|T|}. \end{aligned}$$

□

3 Proof of the main theorem

In this section we prove Theorem 1. The proof is split into three parts depending on the infinite paths that can be found in the tree. One-sided infinite paths are called *rays* and two-sided infinite paths *double rays*. We will discern three types of trees:

1. rayless trees (also called compact trees) are treated in Theorem 3,
2. trees containing rays but no double rays (also called one-ended trees) are treated in Theorem 4, and
3. trees containing at least one double ray are treated in Theorem 6.

For convenience, we will let $\Delta(T)$ denote the least upper bound on the degrees of the vertices in T. Note that if $\Delta(G)$ is infinite, then $\Delta(G) = |G|$ for every connected graph G.

3.1 Compact trees

Our proof for compact trees uses the concept of *rank*, which was introduced by Schmidt in [11] and can be inductively defined as follows.

- Finite trees have rank 0.
- A tree T has rank ρ if
    1. T has not been assigned a rank less than ρ, and if
    2. there is a finite set of S vertices such that each component of $T - S$ has rank less than ρ.

In [11] it was shown that every rayless graph has a rank, and that there is a tree of rank ρ for every ordinal number ρ. We will need the following facts, shown in [11] and [8]. Firstly, there is a canonical choice for the set S in the definition above, by choosing S minimally among all sets that work. This minimal set is called the *core* of T and can be shown to be unique; in particular, it is setwise fixed by every automorphism of T. Secondly, the rank cannot go up by removing additional vertices. In other words, if $S' \subseteq V(T)$ contains the core, then every component of $T - S'$ has rank less than ρ.

Note that the first fact above implies that each rayless tree has a center consisting of either a single vertex or an edge that is preserved by all automorphisms. Just consider the minimal subtree T_S of T containing its core S. This tree T_S is finite, it is preserved by all automorphisms of T, and thus so is its center. Despite the fact that this immediately follows from [11] it was first explicitly stated in [10]. The second fact implies that when we remove all vertices of T_S from T, then every component has strictly smaller rank than T.

Now we state and prove our main result for rayless trees.

Theorem 3. *Let m be an infinite cardinal and let T be a rayless tree with motion m and $\Delta(T) \leq 2^m$. Then $a(T) = 2^{|T|}$.*

Proof. We use transfinite induction on the rank of T. Trees of rank 0 are finite, so they have infinite motion only if they are asymmetric, and therefore $a(T) = 2^{|T|}$ for every tree of rank 0 that satisfies the conditions of the theorem.

For the induction step, let ρ be any ordinal, let T be a tree of rank ρ, and assume that the statement of the theorem holds for any tree with rank $\sigma < \rho$. Let T_S be the minimal subtree containing the core of T. If T has a central vertex, then let w be this central vertex. Otherwise, let w be one endpoint of the central edge. Consider the rooted tree (T, w).

We claim that $a(x) = 2^{|T^x|}$ for every vertex x of (T, w). For vertices not contained in T_S this is true by the induction hypothesis. Now assume that there is a vertex x which does not satisfy the claim and let y be such a vertex at maximal distance from w; note that the maximal distance is finite because T_S is finite. All children of y satisfy the claim, and by Lemma 2 the claim is satisfied for y as well.

If T has a central vertex w, then the statement of the theorem follows immediately from the fact that $a(w) = a(T)$. If there is a central edge ww', then there are $a(w)$ asymmetrizing sets of (T, w) which do not contain w' because the complement of an asymmetrizing set is again asymmetrizing. Clearly any such set is asymmetrizing for T because its stabiliser must fix both w and w'. □

3.2 One-ended trees

We now turn to the case of one-ended trees, that is, trees containing a ray, but no double ray. We invoke a theorem of Polat [7] to prove the following theorem.

Theorem 4. *Let T be a one-ended tree, and let m be an infinite cardinal. If $m(T) = m$ and $\Delta(T) \leq 2^m$, then $a(T) = 2^{|T|}$.*

Proof. Let T satisfy the assumptions of the theorem. Then it contains a ray R. For a vertex x of R we denote by T^x the component of $T - E(R)$ which contains x. We consider T^x as a rooted tree with root x and set $a(x) = a(T^x, x)$. Note that T^x is necessarily rayless, and thus $a(x) = 2^{|T^x|}$ by Theorem 3. Combining this observation with [7, Theorem 3.1], we get

$$a(T, w_0) = \prod_{x \in V(R)} a(x) = \prod_{x \in V(R)} 2^{|T^x|} = 2^{\sum_{x \in V(R)} |T^x|} = 2^{|T|}.$$

If w_0 is fixed by every element of $\mathrm{Aut}(T)$, then $a(T) = a(T, w_0)$. If not, then $a(T) = a(T, w_0)$ by [7, Corollary 3.2]. □

3.3 Trees with double rays

Finally, we consider the case where T is a tree containing double rays. For such a tree T, we let T_* be the tree induced by all vertices that lie on some double ray. For a vertex x of T_* we denote by T^x the component of $T - E(T_*)$ which contains x. We consider T^x as a rooted tree with root x and set $a(x) = a(T^x, x)$. As above, note that T^x is necessarily rayless, and thus $a(x) = 2^{|T^x|}$ by Theorem 3.

Pick an arbitrary root w of T_* and define the concepts of parents, siblings, twins and $\tau(x)$ as in Section 1, in particular recall that $\tau(x)$ is the number of twins of x. The following theorem is the equivalence between conditions (ii), (iv) and (v) of Theorem 3.5 in [7].

Theorem 5. *A tree T which contains double rays is asymmetrizable if and only if $\tau(x) \leq \prod_{y \geq x} a(y)$ for every vertex x of T_*. Moreover, in this case $a(T) = \prod_{x \in T_*} a(x)$.*

We use it to prove our main result for trees containing double rays.

Theorem 6. *Let T be a tree of infinite motion m, and assume that $\Delta(T) \leq 2^m$. If T contains a double ray, then $a(T) = 2^{|T|}$.*

Proof. Pick an arbitrary root in T_*. By Theorem 5 above, it suffices to show that $\tau(x) \leq \prod_{y \geq x} a(y)$ for every vertex x of T_*. Note that if x and x' are twins, then there is an automorphism of T which swaps x and x' and only moves vertices in

$$\bigcup_{y \geq x} T^y \cup \bigcup_{y \geq x'} T^y.$$

If $|\bigcup_{y \geq x} T^y| < m$, then such an automorphism would move fewer than m vertices, hence in this case $\tau(x) = 1$ which is less or equal than $\prod_{y \geq x} a(y)$, because all factors in the product are non-zero.

Hence we may assume that $|\bigcup_{y \geq x} T^y| \geq m$. Note that $\tau(x) \leq \Delta(T) \leq 2^m$. If there is some $y_0 \geq x$ such that $|T^{y_0}| \geq m$, then

$$\tau(x) \leq 2^m \leq a(y_0) \leq \prod_{y \geq x} a(y).$$

If there is no such y, then $a(y) = 2^{|T^y|}$ for every $y \geq x$, and thus

$$\tau(x) \leq 2^m \leq 2^{|\bigcup_{y \geq x} T^y|} = 2^{\sum_{y \geq x} |T^y|} = \prod_{y \geq x} 2^{|T^y|} = \prod_{y \geq x} a(y).$$

By the second part of Theorem 3 we conclude that

$$a(T) = \prod_{x \in T_*} a(x) = \prod_{x \in T_*} 2^{|T^x|} = 2^{\sum_{x \in T_*} |T^x|} = 2^{|T|}. \qquad \square$$

4 Tree-like graphs

We now apply our results to *tree-like graphs*. In [5], they are defined as rooted graphs (G, w), in which each vertex y has a neighbor x such that y is on all shortest x, w-paths. By [5, Theorem 4.2] each tree-like graph G with $\Delta \leq 2^{\aleph_0}$ is asymmetrizable (as an unrooted graph). We present a proof of a strengthened version of this result.

In the terminology of the present paper we could also have defined tree-like graphs as rooted graphs (G, w) in which each vertex has a child of which it is the only parent. Note that w is the only vertex of (G, w) that may have degree 1.

Lemma 7. *Let T be a tree of infinite motion m and $\Delta(T) \leq 2^m$. If each vertex of T is on a double ray, then T has $2^{|T|}$ asymmetrizing sets S in which each vertex of S is adjacent to a vertex of $V(T) \setminus S$, and to any $w \in V(T)$ there are $2^{|T|}$ asymmetrizing sets S_w where w is the only vertex of S_w with no neighbor in $V(T) \setminus S_w$.*

Proof. Form a new tree T' from T by choosing an arbitrary vertex $w \in V(T)$ and by subsequently contracting all edges ab to single vertices if $d_T(w, a)$ is even and $d_T(w, b) = d_T(w, a) + 1$.

T' has motion m, $\Delta(T') \leq 2^m$, and $|T'| = |T|$. By Theorem 1 $a(T') = 2^{|T'|} = 2^{|T|}$. From $a(T', w) \geq a(T')$ and $a(T, w) \geq a(T)$ we also infer that $a(T', w) = a(T, w) = 2^{|T|}$.

Let $\alpha \in \text{Aut}(T, w)$ and α' its restriction to $V(T')$. Then $\alpha' \in \text{Aut}(T', w)$, and α is uniquely determined by its action on (T', w), because each vertex of (T', w) different from w has only one parent. This implies that a set $S \subseteq V(T)$ asymmetrizes (T, w) if $S' = S \cap V(T')$ asymmetrizes (T', w).

Given an asymmetrizing set S' of (T', w) we extend it in two ways to an asymmetrizing set S of (T, w). The first is to set $S = S'$. Clearly this implies that each vertex of S has a neighbor in $V(T) \setminus S$ and, because $a(T', w) = 2^{|T|}$, the number of these set is $2^{|T|}$.

The second way is to form as set S by adding all children of w to S', unless there is a child u of w all of whose children are in S', then we only add u. Note that in this case w has a neighbor that is not in S, because T has no vertices of degree 1. Clearly we obtain $2^{|T|}$ asymmetrizing sets in this way, and either w or u are the only vertices of S all of whose neighbors are in S. □

Theorem 8. *Let G be a tree-like graph with $\Delta(G) \leq 2^{\aleph_0}$. Then $a(G) = 2^{|G|}$.*

Proof. Let G be a tree-like graph (G, w). Then the set of edges yx, where y is on all shortest x, w-paths, are the edges of a spanning subgraph, say F. Clearly F is a forest with no finite components. Each component U has motion $\aleph_0$, unless it is asymmetric. Because $\Delta(U) \leq \Delta(G) \leq 2^{\aleph_0}$ we infer by Theorem 1 that $a(U) = 2^{|U|} \geq 2^{\aleph_0}$.

Note that, by the definition of F, any automorphism α of G that fixes w preserves the components of F. This is the case when w has degree 1.

We now asymmetrize the components of F under the following restrictions. Let U_w be the component of F that contains w. If w has degree 1 we admit any asymmetrizing set S of U_w, otherwise we only admit asymmetrizing sets S in which there is only vertex that has no neighbors in $V(U_w) \setminus S$. For all other components U of F we only admit asymmetrizing sets S where each vertex has a neighbor in $V(U) \setminus S$.

Because F has at most $2^{\aleph_0}$ components we can asymmetrize them pairwise inequivalently with admitted asymmetrizing sets S_U. That is, we asymmetrize by sets S_U that obey the above restrictions, and if U, U' are isomorphic components of F, then there is no isomorphism from U to U' that maps S_U into $S_{U'}$.

Let S be the union of all S_U in such a selection. If w has degree 1, then each $\alpha \in \mathrm{Aut}(G)$ fixes w. Otherwise each $\alpha \in \mathrm{Aut}(G)$ that preserves S fixes w because w is the only vertex of S that has no neighbors in $V(G) \setminus S$. Hence, in either case each $\alpha \in \mathrm{Aut}(G)$ that preserves S also preserves F. As each S_U asymmetrizes U, the set S asymmetrizes F, and thus also G, because F is a spanning subgraph of G.

Let α be the least ordinal of the same cardinality as the set of components of F. By transfinite induction with respect to the well ordering of the components of F induced by α, it is easily seen that there are $2^{|G|}$ asymmetrizing sets S of F. □

Problem 9. *Let m be an uncountable cardinal and G be a tree-like graph with $m(G) = m$ and $\Delta(G) \leq 2^m$. Is G asymmetrizable?*

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