Extending cycles locally to Hamilton cycles

Matthias Hamann Florian Lehner* Julian Pott

November 26, 2013

Abstract

A Hamilton circle in an infinite graph is a homeomorphic copy of the unit circle S^1 that contains all vertices and all ends precisely once. We prove that every connected, locally connected, locally finite, claw-free graph has such a Hamilton circle, extending a result of Oberly and Sumner to infinite graphs. Furthermore, we show that such graphs are Hamilton-connected if and only if they are 3-connected, extending a result of Asratian. *Hamilton-connected* means that between any two vertices there is a Hamilton arc, a homeomorphic copy of the unit interval $[0, 1]$.

1 Introduction

The proofs of many classical sufficient conditions for the existence of a Hamilton cycle can be outlined as follows. Start with an arbitrary cycle, extend the cycle by some additional vertices and iterate this extension procedure until the cycle covers all vertices. It is often the case that the extension happens locally, that is, most of the original cycle—in fact everything outside a bounded distance from some newly added vertex—remains unchanged.

While such a strategy will obviously give a Hamilton cycle for finite graphs, the situation is more complicated with infinite graphs, particularly because it is not entirely clear what an infinite analogue of a Hamilton cycle should be.

Considering spanning rays or spanning double rays as infinite analogues of Hamiltonian cycles has yielded some results (e. g., see Thomassen [23]) but it has the obvious drawback that a graph with more than two ends can never be Hamiltonian. However, there is a different approach suggested by Diestel and Kühn [11, 12]. They define a circle in an infinite graph G to be a homeomorphic image of the unit circle in the end compactification of G. This approach has not only been successful in generalizing Hamiltonicity results to locally finite graphs, it has also yielded generalizations of many theorems about the cycle space (see [9] for an overview).

Unfortunately, the extension strategies mentioned above do not immediately give a Hamiltonian circle in this sense. There will be a limit object if the extension procedure only alters the cycle locally, but it is not guaranteed that this limit object will be a circle. In particular, ensuring injectivity at the ends of G can be challenging. So far, there are some results on Hamilton circles in infinite graphs, see [3, 5, 7, 15, 18]

*The author acknowledges the support of the Austrian Science Fund (FWF), project W1230-N13.

In this paper we present a strategy that ensures that the limit will be a Hamilton circle. We call it a k-local skip-and-glue strategy and it roughly states that every finite 2-regular subgraph can be extended to a larger 2-regular graph with an extension of bounded size, or to a 2-regular graph with fewer components. In Section 3 we define this k-local skip-and-glue strategy and prove that a locally finite graph with such a strategy contains a Hamilton circle.

We then proceed to prove that locally finite claw-free graphs always admit such a strategy. So we shall show that the following two theorems can be extended to locally finite graphs and affirmatively answers questions of Stein [22, Question 5.1.3] and Bruhn, see Stein [22, Question 5.1.4].

Theorem 1.1. [21, Theorem 1] *Every finite connected locally connected claw free graph on at least three vertices is Hamiltonian.*

Theorem 1.2. [1, Theorem 3.4] *Every finite connected locally connected claw-free graph is Hamilton-connected if and only if it is* 3*-connected.*

We also give some corollaries of the two theorems whose infinite but locally finite counterparts are corollaries to the infinite versions of those theorems. Similar questions on Hamilton circles in infinite graphs are currently investigated by Heuer [17].

2 The topological space $|G|$

Let $G = (V, E)$ be a locally finite graph. A *ray* is a one-way infinite path. Two rays are *equivalent* if they lie eventually int he same component of $G - S$ for every finite vertex set $S \subseteq V$. This is an equivalence relation whose equivalence classes are the *ends* of G. For $S \subseteq V$ and an end ω, let $C(S, \omega)$ be the component of $G-S$ that contains some ray, and hence a tail of every ray, in ω and let $\Omega(S, \omega)$ be the set of ends with at least one ray in $C(S, \omega)$.

The space $|G|$ is a topological space on G with its ends such that it coincides on G with its 1-complex and such that the sets

$$C(S, \omega) \cup \Omega(S, \omega) \cup E'(S, \omega)$$

for all finite $S \subseteq V$ and ends ω form a basis for the open neighbourhoods around each end ω, where $E'(S, \omega)$ is any union of half edges $(z, y]$, one for every edge xy with $x \in S$ and $y \in C(S, \omega)$, with z an inner point of xy. It can be proved (see [10]) that $|G|$ is the Freudenthal compactification [14] of the 1-complex G.

A *standard subspace* of $|G|$ is a subspace of the form $\overline{U} \cup \mathring{F}$, where (U, F) is a subgraph of G. We will need the following two lemmas on standard subspaces:

Lemma 2.1. *A standard subspace of $|G|$ is topologically connected if and only if one of the following statements holds.*

(i) *It contains an edge from every finite cut of G which meets both sides* [8, Lemma 8.5.5].

(ii) *It is arc-connected* [13, Theorem 2.6].

A *circle* in $|G|$ is a homeomorphic image of S^1 and an arc in $|G|$ is a homeomorphic image of $[0, 1]$. A circle that contains every vertex and every end of G is a *Hamilton circle* and an arc whose endpoints are vertices and that contain every vertex and every end of G is a *Hamilton arc*. We call G *Hamilton-connected* if there is a Hamilton arc between each two vertices of G.

The *degree* of an end ω in a standard subspace $X \subseteq |G|$ is the supremum of the cardinalities of sets of vertex disjoint arcs ending in ω. This notion of degree for the whole space $|G|$ coincides with the notion of vertex-degree of G.

Combining [11, Theorem 7.1] and [2, Theorem 5], we obtain the following theorem (see [9, Theorem 2.5]):

Theorem 2.2. *Let G be a locally finite graph and $F \subseteq E(G)$. Then the following statements are equivalent:*

(i) *Every vertex and every end has even degree in $\overline{F}$.*

(ii) *Every finite cuts meets F in an even number of edges.*

The following charachterization of subspaces that are circles comes in extremly handy.

Lemma 2.3. [4, Proposition 3] *A standard subspace X of $|G|$ is a circle if and only if it is topologically connected and every vertex and end of G in X has degree* 2.

3 Skip-and-glue extensions

Let H be a subgraph of a graph G. A vertex $v \in G$ has *depth* $d(v, G-H)$ in H, that is the distance from v to anything outside of H.

Let F be a 2-regular finite sugraph of G. Let P be a path whose end vertices v and w are adjacent in F but that is otherwise disjoint from F. Then the 2-regular subgraph $(F \cup P) - vw$ is the *glue extension* of F *by* P *over* the edge vw. A path $R \subseteq F$ is *skippable* if its two end vertices x and y are adjacent in G but not in F. The edge xy is a *bypass* of R. A 2-regular graph F' obtained from F by successively replacing skippable paths by their bypasses is called a *reduction*.

A glue extension of a reduction F' of F by P that covers F is a *skip-and-glue extension of F by P*. Its *depth* is the maximum depth of any vertex of $F \cap P$ in its component of F. It is a *proper* skip-and-glue extension *via u* if there is a vertex u in $P \setminus F$. A skip-and-glue extension of F via u is *k-local* if its *length*, the length of P, is at most k and u is adjacent to F.

For a finite 2-regular subgraph $F \subseteq G$ with at least two components, a *glue fusion* of F is a 2-regular subgraph F' such that F' coincides with $F - e - f$ on $V(D)$ e, f are edges from distinct components of F and such that the edges in $F' \setminus F$ form two disjoint paths connecting the two end vertices of e with those of f. In particular, the glue fusion F' has less components than F: precisely two components of F are *fused* to one of F'. We call D a *skip-and-glue fusion* of F if it is a glue fusion of a reduction F' of F that covers F. This skip-and-glue fusion is *k-local* if the length of each of the two non-trivial paths in $D \setminus F'$ is at most k. A k-local skip-and-glue fusion is *centred around* a vertex $u \in V(F)$ if each vertex of the two non-trivial paths in $F' \setminus F$ has distance at most k to u and if u lies in one of the two fused components.

A *k-local skip-and-glue strategy (with respect to S)* is a function which assigns to every pair (F, u) of a finite 2-regular subgraph $F \subseteq G$ (covering S) all whose components contain at least k vertices and a vertex $u \in N(F)$ a k-local skip-and-glue extension of F via u and which assigns to every pair (F, u) of a finite 2-regular disconnected subgraph $F \subseteq G$ (covering S) all whose components contain at least k vertices and vertex $u \in V(F)$ that has a neighbour in a different component of F a k-local skip-and-glue fusion of F centred around u.

Note that the image of a pair (F, u) of a k-local skip-and-glue strategy is not necessarily a graph all whose components contain at least k vertices.

Lemma 3.1. *Let G be a locally finite connected graph containing a cycle. If there is a k-local skip-and-glue strategy for G, then G is 2-connected.*

Proof. Let C be a cycle in G and suppose that some vertex v separates G. Let P be a path with one end vertex in C and one end vertex in a component of $G - v$ not containing C. Let C' be a cycle containing C and a maximal number of vertices from P. As there is a k-local skip-and-glue strategy for G, all vertices of P are contained in C'. This contradicts that v is a separating vertex as it does not separate C' but P meets two components of $G - v$. □

Lemma 3.2. *Let G be a locally finite 2-connected graph. Then every neighbourhood of every end contains a cycle of length at least n for all $n \in \mathbb{N}$.*

Proof. For an end ω of G and a neighbourhood U of ω and some $n \in \mathbb{N}$, let R_1, R_2 be two vertex disjoint rays in ω that lie in U, which exist as G is 2-connected [16]. Let $P_1, P_2, \ldots$ be an infinite sequence of vertex disjoint finite R_1–R_2 paths. Such a sequence exists, as in every neighbourhood of ω there is one such path, and for any collection of finitely many paths there is a neighbourhood of ω avoiding those. Let P_i and P_j be two of these path whose end vertices on R_1 have distance at least n. The unique cycle in $R_1 \cup P_j \cup R_2 \cup P_i$ has length at least n. □

We will see later that a locally finite graph which satisfies the conditions of Theorem 1.1 always admits a 4-local skip-and-glue strategy, hence the following lemma can be used to extend the theorem to locally finite graphs.

Theorem 3.3. *Let $k \in \mathbb{N}$ and let $G = (V, E)$ be a locally finite connected graph.*

(I) *If G has a k-local skip-and-glue strategy and contains a cycle of length at least k, then G is hamiltonian.*

(II) *If G contains a v–w path P whose end vertices have depth at least $k+1$ in P and $G + vw$ has a k-local skip-and-glue strategy with respect to $V(P)$, then G contains a v–w arc that is hamiltonian.*

Proof. To prove (I) we will define a sequence of finite cycles $(C_i)_{i \in \mathbb{N}}$ such that

(i) $V(C_{i+1})$ contains $V(C_i)$,

(ii) Every edge with depth at least $2k$ in C_i is contained in C_{i+1} if and only if it is contained in C_i.

(iii) every vertex is contained in some C_i,

(iv) for every end ω of G and every finite set $V' \subseteq V$ there is a finite cut F separating ω from V' and an index i_0 such that $|E(C_{i_0}) \cap F| = 2$ and $E(C_{i_0}) \cap F = E(C_i) \cap F$ for every $i > i_0$.

Let C be the limit of the sequence C_i, that is, C is the set of all those edges that are contained in the C_i eventually. Let us first show that $\overline{C}$ is a Hamilton circle as soon as (i) to (iv) are satisfied.

Together, (i) and (iii) imply $\bigcup V(C_i) = V(G)$. By (iii) and as G is locally finite there is for every vertex v an index j such that C_j contains all vertices with distance $k+1$ or less from v. By (ii) every vertex has degree 2 in C and $V(C) = V(G)$. Let F be a finite cut with bipartition (A, B). Then there is some $j \in \mathbb{N}$ such that all vertices of F lie in C_j. As C_{j+1} is connected and meets both A and B, it must contain an even number of edges from F. These edges are contained in C by (ii). As C meets every finite cut in an even number of edges, every vertex and every end of $\overline{C}$ has even degree by Theorem 2.2 and $\overline{C}$ is topologically connected by Lemma 2.1 (i). Additionally, Lemma 2.1 (ii) implies that the standard subspace $\overline{C}$ is arc-connected and hence its degree is at least 2. We already saw that every vertex has degree 2. Since every vertex lies in $\overline{C}$, so does every end. By (iv) we find for every end ω a sequence of cuts $(F_i)_{i\in\mathbb{N}}$ such that the components of $G - F_i$ that contain ω converge to ω and such that each of these cuts contains precisely two edges of C. Thus, the degree of ω is at most 2 and hence precisely 2. This implies that the standard subspace $\overline{C}$ is a circle by Lemma 2.3. It is a Hamilton circle, as it contains every vertex and every end.

We define the C_i recursively. Besides this sequence, we shall define a second sequence $(\Lambda_i)_{i\in\mathbb{N}}$ of labellings $\Lambda_i \colon V \to \mathbb{N}$. For $i, q \in \mathbb{N}$ let

$$X^i_q := \{v \in V \mid \Lambda_i(v) = q\}.$$

Then the sequences will satisfy the following properties for every $1 \leq q$:

(Λ1) X^i_0 is finite.

(Λ2) $C_i[X^i_q]$ is a subpath of C_i.

(Λ3) Any non-empty cut $E(X^i_q, V \setminus X^i_q)$ intersects with C_i precisely twice.

(Λ4) Every vertex in X^i_q not in C_i has distance more than $2k$ from $V \setminus X^i_q$.

(Λ5) Every vertex of C_i of depth $2k$ or more in C_i lies in X^i_0.

Let C_0 be any finite cycle of length at least k, which exists by assumption. For $j \in \mathbb{N}$ assume that all C_i and Λ_i with $i \leq j$ have been defined and that they satisfy (i), (ii), and (Λ2) to (Λ5). Let $B(i) \subseteq G$ for $i \in \mathbb{N}$ be the restriction of G to the vertices with distance at most i from C_j. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be the set of infinite components of $G \setminus B(2k)$. Let $\mathcal{D}_f$ be the set of finite components of $G \setminus B(2k)$. Due to Lemma 3.1, we know that G is 2-connected. So according to Lemma 3.2, we find a cycle within every component $D_\ell \in \mathcal{D}$ of length at least k that has distance at least $2k$ to the boundary of D_ℓ, adding successively all vertices with distance at most k to this initial cycle via k-local skip-and-glue extensions we have a cycle C^ℓ in D_ℓ with at least k vertices of depth at least k in C^ℓ. Let $\mathcal{C} = \{C^1, \ldots, C^n\}$. Let $x_1, \ldots, x_m$ be an enumeration of some vertices of $\bigcup \mathcal{D}$ with

$$\left(B\left((n+5)\,k\right) \cap \bigcup \mathcal{D}\right) \cup \bigcup \mathcal{C} \subseteq G[x_1, \ldots, x_m]$$

such that for every $i \leq m$ each component of $G[x_1, \dots, x_i]$ has a vertex in some C^ℓ. By our k-local skip-and-glue strategy, there is a sequence

$$C_j \cup \bigcup \mathcal{C} = F_0, \dots, F_m = F$$

of finite 2-regular graphs such that $F_{i+1} = F_i$ if $x_{i+1} \in V(F_i)$, and such that F_{i+1} is obtained from F_i by a k-local skip-and-glue extension via x_{i+1} if $x_{i+1} \notin V(F_i)$. Note that all these extensions have length at most k and thus $C_j \subseteq F$. Clearly, F has at most $n+1$ components as F_0 has precisely $n+1$. Furthermore, the choice of the vertices x_i gives us that the vertices of F in D_i induce a connected graph in G for each $i \leq n$.

We proceed with a finite sequence $F = F^0, \dots, F^p = F'$ of k-local skip-and-glue fusions: If some vertex u of $F^i \cap \bigcup \mathcal{D} \subseteq G - B(2k)$ has a neighbour in a different component of F^i, let F^{i+1} be a k-local skip-and-glue fusion of F^i centred around u. Since the number of components of F is at most $n+1$ and reduces by at least 1 with each skip-and-glue fusion we have $p \leq n+1$. Then we have the following properties:

(1) Every $D_i \in \mathcal{D}$ contains vertices from exactly one component of F' and all vertices within $B((n+5)k) \cap D_i$ lie in this component.

(2) $C_j = F' \cap B(m)$

(3) F' has at most $n+1$ components.

Next, we construct a finite sequence of k-local skip-and-glue extensions via vertices in $B((n+5)k) \cup \bigcup \mathcal{D}_f$ such that its first element is F' and its last element E contains every vertex of $B((n+5)k) \cup \bigcup \mathcal{D}_f$. As every extension has length at most k, we have the following properties:

(4) All vertices with depth at least k in C_j lie in a common component of E.

(5) Only one component of E meets $D_i \setminus B(3k)$ for every $i \leq n$.

(6) $V(E)$ is connected in G.

(7) E has at most $n+1$ components.

Let $E = E_0, \dots, E_\ell = C_{j+1}$ be a sequence of 2-regular subgraphs of G such that E_ℓ is connected and E_{i+1} is a k-local skip-and-glue fusion centred around a vertex in the unique component U_i of E_i that contains all vertices with depth at least k in C_j. Note that this is well-defined as $V(U_i)$ is contained in the fused cycle U_{i+1}. The properties (i), (ii), and (iii) are direct consequences of the construction of C_{j+1} as $V(B((n+5)k)) \subseteq V(C_{j+1})$.

To proof (iv) let us define a finite sequence $\Lambda^0, \dots, \Lambda^\ell = \Lambda_{j+1}$ of labellings $\Lambda^i \colon V \to \mathbb{N}$ (with the same ℓ as above). For $i, q \in \mathbb{N}$ let

$$Y_q^i := \{v \in V \mid \Lambda^i(v) = q\}.$$

We shall construct these labellings such that they satisfy the following conditions for $i \in \mathbb{N}$, $q > 0$, and $1 \leq p \leq \ell$. Note that $\ell \leq n$.

(Λ6) Y_0^i is finite.

(Λ7) $E_i[Y_q^i]$ is either a component of E_i or a subpath of U_i.

(Λ8) The intersection of any non-empty cut $E(Y_q^i, V \setminus Y_q^i)$ with E_i contains either none or precisely two edges.

(Λ9) Every vertex in Y_q^i not in E_i has distance more than $(n+2-p)k$ from $V \setminus Y_q^i$.

(Λ10) Every vertex of C_j of depth $2k$ or more in C_j lies in Y_0^i.

Let $K_0, \dots, K_p$ be the components of E_0 such that K_0 contains all vertices with depth at least k in C_j, cp. (4). Let us define a labelling $\Lambda^0 \colon V \to \mathbb{N}$. For a vertex $x \in V(K_i)$, let $\Lambda^0(x) = i$. For every component $D_i \in \mathcal{D}$, there is a unique label $\ell \in \mathbb{N}$ with $\Lambda^0(x) = \ell$ for all $x \in V(D_i \setminus B(3k))$, cp. (5). Let Λ^0 map every $y \in V(D_i \setminus E)$ to ℓ. Clearly, Λ^0 satisfies (Λ7) to (Λ10).

Assume Λ^p has been defined for $0 \leq p < \ell$. The cycle U_{p+1} is the union of a reduction R_p of U_p, a reduction R of some other component U of E_p and two disjoint paths P, Q minus two edges $e_p \in E(R_p)$ and $e \in E(R)$. If both end vertices of e_p have the Λ^p-label 0 let λ be the unique label of the vertices of U. Otherwise, let λ be the largest Λ^p-label at an end vertex of e_p. Let W be the set consisting of the vertices on P, Q, and R as well as all the vertices labelled with the label of U. Let Λ^{p+1} equal Λ^p outside of W and let Λ^{p+1} map every vertex of W to λ.

Clearly, properties (Λ6) and (Λ7) are kept throughout the construction. Moreover, (Λ8) is a direct consequence of (Λ7). Since all fusions are k-local we have (Λ9). Every vertex of depth at least k in C_j lies in all U_i and thus the fusions are centred around vertices with depth at most k in C_j as those vertices have neighbours outside of U_i. Since every vertex with depth $2k$ in C_j lies on the reduction of U_i for all k-local fusions of E_i, we have (Λ10). Thus we have (Λ1) to (Λ5) as $E_\ell = C_{j+1}$ is connected, $\Lambda^\ell = \Lambda_{j+1}$, and $\ell \leq n$.

It remains to prove (iv). Let us consider any finite set $S \subseteq V$ and any end ω of G. By (i) and (iii), we find an index j such that every vertex of S has depth more than $2k$ in C_j. So by (Λ5), it lies in X_0^{j+1}. The end ω lies in some infinite component K of $G - B((n+5)k)$. By (Λ4), all vertices of K have the same Λ_{j+1}-label q. As $\mathcal{B} = E(X_q^{j+1}, V \setminus X_q^{j+1})$ is non-empty, it intersects with C_{j+1} precisely twice due to (Λ3). By (Λ4), the vertices incident with $\mathcal{B}$ have depth $2k$ or more in C_{j+1}. So (ii) implies that $\mathcal{B}$ meets C_i in precisely these two edges for every $i \geq j+1$. This shows (iv) and completes the proof of (I).

To prove (II) we pick a sequence $P + vw = F_0, \dots, F_n$ of cycles such that F_{i+1} is a k-local skip-and-glue extension of F_i and F_n covers every vertex of distance at most $2k+1$ from vw. Clearly, vw is an edge of F_n. Following the proof of (I) there is a sequence $F_n = C_0, C_1, \dots$ of finite cycles satisfying (i) to (iv). Thus for their limit C its closure $\overline{C}$ is a Hamilton circle and C contains vw by (ii). This completes the proof of (II) as $C - vw$ is a Hamilton arc of G with end vertices v and w. □

4 Locally connected graphs

For a subgraph F of a graph G, let us call a vertex $x \in V(F)$ *skippable* if x has degree 2 in F and its two neighbours y, z are adjacent in G but not in F. We call yz an *x-bypass*. Note that, if F is 2-regular, x is skippable if and only if yxz is a skippable path in F.

Lemma 4.1. *Every connected locally connected claw-free graph G has a 5-local skip-and-glue strategy.*

Furthermore, for $x, y \in V(G)$ the graph $G + xy$ has a 5-local skip-and-glue strategy with respect to $V(P)$ for some x–y path P that contains all vertices of distance at most 3 from x or y.

Proof. Let $G = (V, E)$ be a locally finite connected locally connected claw-free graph on at least three vertices. Let F be a finite subgraph of G with maximum degree 2 all whose components contain at least four vertices. Let C be any component of F. Note that C is either a cycle or a path. If C is a path we require its end vertices to have depth at least 3 in C and add the edge between those end vertices to G. Note that in the later construction, none of these new edges appears as they are to deep in their respective component. For the readability of the proof we omit their presence from now on and consider G to be the graph with these edges added and F to be 2-regular. With a slight stretch of terminology, we consider G to be claw-free, although it is G without these additional edges that is claw-free.

On each component C we choose one order $\leq_C$ of the two available canonical cyclic orders. For every vertex $v \in V(C)$ denote by v^- its predecessor with respect to $\leq_C$ and by v^+ its successor.

Let $u \notin V(F)$ be a vertex of G that has a neighbour v on F. To show that there is a k-local skip-and-glue strategy we have to provide a k-local skip-and-glue extension of F via u. As G is locally connected $N(v)$ is connected; it contains u, v^-, and v^+. Thus there is a shortest u–$\{v^-, v^+\}$ path $P = p_0p_1 \dots p_k$ with all its vertices in $N(v)$. We may assume that $p_0 = u$ and $p_k = v^+$ by choosing the other canonical order for the component containing u if necessary. Clearly, P does not contain v^-. By minimality P is induced and thus if $k \geq 4$, we have the claw $G[v, p_0, p_2, p_4]$ in G. Hence $k \leq 3$ and the length of P is at most 3.

If all inner vertices of P which lie on F are skippable and no two consecutive vertices of P are adjacent on F, then we can replace every skippable path $p_i^- p_i p_i^+$ by the edge $p_i^- p_i^+$. Furthermore, we replace the edge vv^+ by the path $vuPv^+$ to obtain a 4-local skip-and-glue extension of F via u. This covers the case that u is adjacent to v^- or v^+.

Thus we may assume that P has an inner vertex and since G is claw-free it holds that $v^-v^+ \in E(G)$. Since the component of F containing v has at least four vertices, v is skippable. If some inner vertex p_i of P lies on F and is not skippable, then the path $p_i^- p_i p_i^+$ is not skippable. Thus, we have $p_i^- p_i^+ \notin E$.

It remains to construct a k-local skip-and-glue extension if either some vertex of P is not skippable or some edge of P lies on F. Let p_i be the first vertex on P that is either not skippable or incident with an edge of $P \cap F$. As $G[v, p_i, p_i^-, p_i^+]$ is not a claw, there is an edge vp_i^- or vp_i^+ in the first case. As P lies in the neighbourhood of v, there is an edge vp_i^- or vp_i^+ in the second case, too. As v is skippable, we reduce F by replacing v^-vv^+ by the edge v^-v^+ and $p_j^- p_j p_j^+$ by the edge $p_j^- p_j^+$ for all $j < i$ where p_j is skippable. If v and p_i^- are adjacent, then we replace the edge $p_ip_i^-$ by the path $p_i^- vuPp_i$. Otherwise, v and p_i^+ are adjacent and we replace the edge $p_ip_i^+$ by the path $p_iPuvp_i^+$. Note that $i \leq 2$. Thus, we obtain a 4-local skip-and-glue extension of F via u in both cases.

Next, let us assume that F has more than one component each containing at least four vertices. Let $v \in V(F)$ be a vertex in a component K of F with a neighbour in some other component of F. Following the above argument for each vertex $u \in N(v) \cap V(F \setminus K)$ we have a 4-local skip-and-glue extension K' of K by P via u over some edge $e \in E(K)$ such that the inner vertices of P lie in $N(v) \cup \{v\}$ and u is the only vertex in P from $F \setminus K$.[1]

Similarly, there is a path Q from $\{u^-, u^+\}$ to one of the two neighbours of u on K', which are its neighbours on P.

For the following construction let us choose such P and Q with some minimality conditions:

(i) Let P be shortest possible.

(ii) With respect to (i) let Q be shortest possible.

Clearly, $|P| = 2$ if and only if there is an edge e in K with both its end vertices adjacent to u. Note that no inner vertex of Q lies on P, as this is in contradiction to the minimality of P.

Suppose that there is an inner vertex q of Q in K' that is either not skippable or incident with an edge of Q that lies in K'. As seen above $q \in V(K) \setminus V(P)$. Let a, b be the two neighbours of q on K'. As $G[u, q, a, b]$ is not a claw, there is an edge ua or ub if q is not skippable. As Q lies in the neighbourhood of u, there is an edge ua or ub if q is incident with an edge of Q on K', too. By symmetry we may assume that a is adjacent to u. If $qa \in E(K)$ we have a contradiction to the minimality of Q as we could have chosen $P = qua$ and shortened Q to end in q. Thus we may assume that $qa \in E(K') \setminus E(K)$. By the construction of K' the edge qa is a z-bypass for some $z \in V(P)$. This contradicts the minimality of P as $quPz$ is shorter since both u and z are inner vertices of P.

Thus every inner vertex of Q in K' is skippable and no edge of Q is an edge of K'.

If all inner vertices of Q are skippable in F and in K' and no edge of Q lies in $F \setminus K$ or K', let F' be the reduction of F where the inner vertices of Q and P in F are replaced by their bypasses and let L be the set consisting of the edge e and the edges from u to the end vertices of Q. Then $(F' \cup P \cup Q) - L$ is a skip-and-glue fusion of F. It is 5-local as Q has length at most 3 and every component of $P - L$ has length at most 2. Clearly, it is centred around v.

Thus we may assume that there is an inner vertex on Q in $F \setminus K$ that is not skippable or incident with an edge of $Q \cap (F \setminus K)$. First note that every vertex of Q is adjacent to a neighbour of u in F if u is not skippable in F. This implies, by the minimal choice of Q and as Q contains an inner vertex, that u is skippable in F.

Let q be the last inner vertex of Q in $F \setminus K$ that is not skippable or incident with an edge of $Q \cap (F \setminus K)$. Thus its subpath qQp with $p \in V(P)$ does not contain any such vertices. Let a, b be the two neighbours of q on F. As $G[u, q, a, b]$ is not a claw, there is an edge ua or ub if q is not skippable. As Q lies in the neighbourhood of u, there is an edge ua or ub if q is incident with an edge of Q on F, too. By symmetry we may assume that a is adjacent to u.

Let F' be the reduction of F where the inner vertices of Q and P in F are replaced by their bypasses and let $L = \{e, up, aq\}$. Then $(F' \cup P \cup qQp) - L + au$ is a skip-and-glue fusion of F. It is 4-local as qQp has length at most 2 and the

[1]Indeed, we get a path P completely contained in $N(v) \cup \{v\}$, and such a path of minimal length contains only one vertex from $F \setminus K$. We forget that its end vertices are contained in $N(v) \cup \{v\}$ for later convenience.

component of $P-up$ containing p has length at most 2 and the other component of $P-up$ has length at most 3. Clearly, it is centred around v. □

The following is a corollary extracted from the previous proof.

Corollary 4.2. *Every connected locally connected locally finite claw-free graph on at least k vertices contains a cycle of length at least k.* □

Now we can combine our previous results to prove our first main theorem.

Theorem 4.3. *Every connected locally connected locally finite claw-free graph on at least three vertices has a Hamilton circle.*

Proof. Let G be a connected locally connected locally finite claw-free graph on at least three vertices. By Corollary 4.2 the graph G contains a cycle of length $\min\{|G|, 5\}$. Due to Lemma 4.1, there is a 5-local skip-and-glue strategy for G. Thus, the assertion is a direct consequence of Theorem 3.3. □

Let us now turn our attention to Hamilton arcs. We deduce from the proof of Theorem 1.2 in [1] the following proposition that is valid for all locally finite graph.

Proposition 4.4. *Let G be a connected locally connected locally finite claw-free graph. For each two $x, y \in V(G)$ that do not disconnect G there is an x-y path of length at least 3 that contains $N(x) \cup N(y)$.* □

In the proof that the result of Proposition 4.4 implies that the graph has a Hamilton arc, Asratian showed in [1] the following:

Proposition 4.5. *Let G be a connected locally connected locally finite claw-free graph. For each two $x, y \in V(G)$ that do not disconnect G there is an x-y path of length at least 3 that contains all vertices of distance at most 2 to either x or y.* □

Using our terminology, Proposition 4.5 implies the existence of some x–y path P such that the depth in P of its end vertices is at least 3. This enables us to prove our second main theorem.

Theorem 4.6. *In every connected locally connected locally finite claw-free graph on at least 3 vertices, any two vertices that do not disconnect the graph are connected by a Hamilton arc.*

Proof. Let G be a connected locally connected locally finite claw-free graph and let $x, y \in V(G)$ be distinct vertices that do not disconnect G. As mentioned before, Proposition 4.5 implies that we find an x–y path P such that x and y have depth 3 in P. Since $G+xy$ has a 5-local skip-and-glue strategy with respect to $V(P)$ by Lemma 4.1 there is a sequence $P + xy = C_0, \ldots, C_n$ of cycles in $G + xy$ such that C_{i+1} is a 5-local skip-and-glue extensions of C_i and such that C_n contains all vertices of distance at most 6 from x or y. Thus the assertion follows from Theorem 3.3. □

As a corollary of the previous theorem, we obtain the following theorem:

Theorem 4.7. *A connected locally connected locally finite claw-free graph on at least four vertices is Hamilton-connected if and only if it is 3-connected.* □

5 Further sufficient conditions for the existence of a Hamilton circle

In this section, we deduce some corollaries from the main theorems of Section 4. To shorten this section, we say that a graph G satisfies $(\star)$ if the following statements are true:

(i) G has a Hamilton circle.

(ii) For each two vertices $u, v \in V(G)$ that do not separate G, there is a Hamilton u–v arc in $|G|$.

(iii) G is Hamilton-connected and $|V(G)| \geq 4$ if and only if it is 3-connected.

It is well-known that line graphs are claw-free. Thus, we directly obtain the following corollary (whose finite version for Hamilton cycles is due to Oberly and Sumner [21, Corollary 1]):

Corollary 5.1. *Let G be a locally finite connected locally connected line graph on at least three vertices. Then G satisfies $(\star)$.* □

The proof that the assumptions of the following corollary imply that $L(G)$ is locally connected is the same as for finite graphs. Thus, we obtain the following corollary (whose finite version for Hamilton cycles is due to Oberly and Sumner [21, Corollaries 2 and 3]):

Corollary 5.2. *Let G be a locally finite connected graph on at least three vertices such that either every edge lies on a triangle or G is locally connected. Then $L(G)$ satisfies $(\star)$.* □

For the following two corollaries the proofs that their assumptions imply that $L(L(G))$, $L(G^2)$, respectively, is locally connected is the same as for finite graphs, see [21, Corollaries 4 and 5]. The finite version for Hamilton cycles of Corollary 5.3 is due to Chartrand and Wall [6]) and that of Corollary 5.4 is due to Nebeský [20].

Corollary 5.3. *Let G be a locally finite connected graph with minimum degree at least 3. Then $L(L(G))$ satisfies $(\star)$.* □

Note that for a graph G, its *square* G^2 has $V(G)$ as its set of vertices and two distinct vertices are adjacent in G^2 if their distance in G is at most 2.

Corollary 5.4. *Let G be a locally finite connected graph on at least three vertices. Then $L(G^2)$ satisfies $(\star)$.* □

The last corollary of the results of Section 4 carries over the result by Matthews and Sumner [19, Corollary 1] for Hamilton cycles in finite graphs to locally finite graphs. Again, their proof carries over almost verbally.

Corollary 5.5. *Let G be a locally finite connected graph on at least three vertices. If G^2 is claw-free, then it satisfies $(\star)$.* □

References

[1] A.S. Asratian, *Every 3-connected, locally connected, claw-free graph is Hamilton-connected*, J. Graph Theory **23** (1996), 191–201.

[2] E. Berger and H. Bruhn, *Eulerian edge sets in locally finite graphs*, Combinatorica **31** (2011), 21–38.

[3] R.C. Brewster and D. Funk, *On the hamiltonicity of line graphs of locally finite,* 6*-edge-connected graphs*, J. Graph Theory **71** (2012), 182–191.

[4] H. Bruhn and M. Stein, *On end degrees and infinite circuits in locally finite graphs*, Combinatorica **27** (2007), 269–291.

[5] H. Bruhn and X. Yu, *Hamilton circles in planar locally finite graphs*, Preprint 2005.

[6] G. Chartrand and C.E. Wall, *On the Hamiltonian index of a graph*, Studia Sci. Math. Hungar. **8** (1973), 43–48.

[7] Q. Cui, J. Wang, and X. Yu, *Hamilton circles in infinite planar graphs*, J. Combin. Theory Ser. B **99** (2009), 110–138.

[8] R. Diestel, *Graph Theory*, 4th ed., Springer, 2010.

[9] R. Diestel, *Locally finite graphs with ends: a topological approach*, Discrete Math. **310–312** (2010–11), 2750–2765 (310); 1423–1447 (311); 21–29 (312), arXiv:0912.4213.

[10] R. Diestel and D. Kühn, *Graph-theoretical versus topological ends of graphs*, J. Combin. Theory (Series B) **87** (2003), 197–206.

[11] ______, *On infinite cycles I*, Combinatorica **24** (2004), 68–89.

[12] ______, *On infinite cycles II*, Combinatorica **24** (2004), 91–116.

[13] ______, *Topological paths, cycles and spanning trees in infinite graphs*, Europ. J. Combinatorics **25** (2004), 835–862.

[14] H. Freudenthal, *Über die Enden topologischer Räume und Gruppen*, Math. Zeitschr. **33** (1931), 692–713.

[15] A. Georgakopoulos, *Infinite hamilton cycles in squares of locally finite graphs*, Advances in Mathematics **220** (2009), 670–705.

[16] R. Halin, *A note on Menger's theorem for infinite locally finite graphs*, Abh. Math. Sem. Univ. Hamburg **40** (1974), 111–114.

[17] K. Heuer, *personal communication.*

[18] F. Lehner, *On spanning tree packings of highly edge connected graphs*, arXiv:1109.6787, 2011.

[19] M.M. Matthews and D.P. Sumner, *Hamiltonian results in* $K_{1,3}$*-free graphs*, J. Graph Theory **8** (1984), 139–146.

[20] L. Nebeský, *On the line graph of the square and the square of the line graph of a connected graph*, Časopis Pěst. Mat. **98** (1973), 285–287.

[21] D.J. Oberly and D.P. Sumner, *Every connected, locally connected nontrivial graph with no induced claw is Hamiltonian*, J. Graph Theory **3** (1979), 351–356.

[22] M. Stein, *Extremal infinite graph theory*, Discrete Math. **311** (2011), 1472–1496.

[23] C. Thomassen, *Hamiltonian paths in squares of infinite locally finite blocks*, Annals of Discrete Mathematics **3** (1978), 269–277.