

NON-RECONSTRUCTIBLE LOCALLY FINITE GRAPHS

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ABSTRACT. Two graphs G and H are *hypomorphic* if there exists a bijection $\varphi: V(G) \rightarrow V(H)$ such that $G - v \cong H - \varphi(v)$ for each $v \in V(G)$. A graph G is *reconstructible* if $H \cong G$ for all H hypomorphic to G .

Nash-Williams proved that all locally finite connected graphs with a finite number ≥ 2 of ends are reconstructible, and asked whether locally finite connected graphs with one end or countably many ends are also reconstructible.

In this paper we construct non-reconstructible connected graphs of bounded maximum degree with one and countably many ends respectively, answering the two questions of Nash-Williams about the reconstruction of locally finite graphs in the negative.

1. INTRODUCTION

Two graphs G and H are *hypomorphic* if there exists a bijection φ between their vertex sets such that the induced subgraphs $G - v$ and $H - \varphi(v)$ are isomorphic for each vertex v of G . We say that a graph G is *reconstructible* if $H \cong G$ for every H hypomorphic to G . The *Reconstruction Conjecture*, a famous unsolved problem attributed to Kelly and Ulam, suggests that every finite graph with at least three vertices is reconstructible.

For an overview of results towards the Reconstruction Conjecture for finite graphs see the survey of Bondy and Hemminger [3]. The corresponding reconstruction problem for infinite graphs is false: the countable regular tree T_∞ , and two disjoint copies of it (written as $T_\infty \cup T_\infty$) are easily seen to be non-homeomorphic reconstructions of each other. This example, however, contains vertices of infinite degree. Regarding locally finite graphs, Harary, Schwenk and Scott [7] showed that there exists a non-reconstructible locally finite forest. However, they conjectured that the Reconstruction Conjecture should hold for locally finite trees. This conjecture has been verified for locally finite trees with at most countably many ends in a series of paper [1, 2, 11]. However, very recently, the present authors have constructed a counterexample to the conjecture of Harary, Schwenk and Scott.

Theorem 1.1 (Bowler, Erde, Heinig, Lehner, Pitz [4]). *There exists a non-reconstructible tree of maximum degree three.*

The Reconstruction Conjecture has also been considered for general locally finite graphs. Nash-Williams [8] showed that if $p \geq 3$ is an integer, then any locally finite connected graph with exactly p ends is reconstructible; and in [10] he showed the same is true for $p = 2$. The case $p = 2$ is significantly more difficult. Broadly speaking this is because every graph with $p \geq 3$ ends has some identifiable finite

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‘centre’, from which the ends can be thought of as branching out. A two-ended graph however can be structured like a double ray, without an identifiable ‘centre’.

The case of 1-ended graphs is even harder, and the following problems from a survey of Nash-Williams [9], which would generalise the corresponding results established for trees, have remained open.

Problem 1 (Nash-Williams). *Is every locally finite connected graph with exactly one end reconstructible?*

Problem 2 (Nash-Williams). *Is every locally finite connected graph with countably many ends reconstructible?*

In this paper, we extend our methods from [4] to construct examples showing that both of Nash-Williams’ questions have negative answers. Our examples will not only be locally finite, but in fact have bounded degree.

Theorem 1.2. *There is a connected one-ended non-reconstructible graph with bounded maximum degree.*

Theorem 1.3. *There is a connected countably-ended non-reconstructible graph with bounded maximum degree.*

Since every locally finite connected graph has either finitely many, countably many or continuum many ends, Theorems 1.1, 1.2 and 1.3 together with the results of Nash-Williams provide a complete picture about what can be said about number of ends versus reconstruction:

- A locally finite tree with at most countably many ends is reconstructible; but there are non-reconstructible locally finite trees with continuum many ends.
- A locally finite connected graph with at least two, but a finite number of ends is reconstructible; but there are non-reconstructible locally finite connected graphs with one, countably many, and continuum many ends respectively.

This paper is organised as follows: In the next section we give a short, high-level overview of our constructions which answer Nash-Williams’ problems. In Sections 3 and 4, we develop the technical tools necessary for our construction, and in Sections 5 and 6, we prove Theorems 1.2 and 1.3.

For standard graph theoretical concepts we follow the notation in [5].

2. SKETCH OF THE CONSTRUCTION

In this section we sketch the main ideas of the construction in three steps. First, we quickly recall our construction of two hypomorphic, non-isomorphic locally finite trees from [4]. We will then outline how to adapt the construction to obtain a one-ended-, and a countably-ended counterexample respectively.

2.1. The tree case. This section contains a very brief summary of the much more detailed sketch from [4]. The strategy is to build trees T and S recursively, where at each step of the construction we ensure for some new vertex v already chosen for T that there is a corresponding vertex w of S with $T - v \cong S - w$, or vice versa. This will ensure that by the end of the construction, the trees we have built are hypomorphic.

More precisely, at step n we will construct subtrees T_n and S_n of our eventual trees, where some of the leaves of these subtrees have been coloured in two colours, say red and blue. We will only further extend the trees from these coloured leaves, and we will extend from leaves of the same colour in the same way. We also make sure that earlier partial isomorphisms between $T_n - v_i \cong S_n - w_i$ preserve leaf colours. Together, these requirements guarantee that earlier partial isomorphisms always extend to the next step.

The T_n will be nested, and we will take T to be the union of all of them; similarly the S_n will be nested and we take S to be the union of all of them. To ensure that T and S do not end up being isomorphic, we first ensure, for each n , that there is no isomorphism from T_n to S_n . Our second requirement is that T or S beyond any coloured leaf of T_n or S_n begins with a long non-branching path, longer than any such path appearing in T_n or S_n . Together, this implies that T and S are not isomorphic.

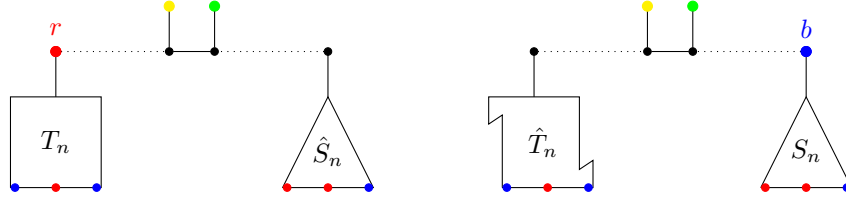


FIGURE 1. A first approximation of T_{n+1} on the left, and S_{n+1} on the right. All dotted lines are long non-branching paths.

Algorithm Stage One: Suppose now that we have already constructed T_n and S_n and wish to construct T_{n+1} and S_{n+1} . Suppose further that we are given a vertex v of T_n for which we wish to find a partner w in S_{n+1} so that $T - v$ and $S - w$ are isomorphic. We begin by building a tree $\hat{T}_n \not\cong T_n$ which has some vertex w such that $T_n - v \cong \hat{T}_n - w$. This can be done by taking the components of $T_n - v$ and arranging them suitably around the new vertex w .

We will take S_{n+1} to include S_n and \hat{T}_n , with the copies of red and blue leaves in \hat{T}_n also coloured red and blue respectively. As indicated on the right in Figure 1, we add long non-branching paths to some blue leaf b of S_n and to some red leaf r of \hat{T}_n and join these paths at their other endpoints by some edge e_n . We also join two new leaves y and g to the endvertices of e_n . We colour the leaf y yellow and the leaf g green. To ensure that $T_{n+1} - v \cong S_{n+1} - w$, we take T_{n+1} to include T_n together with a copy \hat{S}_n of S_n , with its leaves coloured appropriately, and joined up in the same way, as indicated on the left in Figure 1. Note that, whilst \hat{S}_n and S_n are isomorphic as graphs, we make a distinction as we want to lift the partial isomorphisms between $T_n - v_i \cong S_n - w_i$ to these new graphs, and our notation aims to emphasize the natural inclusions $T_n \subset T_{n+1}$ and $S_n \subset S_{n+1}$.

Algorithm Stage Two: We now have committed ourselves to two targets which are seemingly irreconcilable: first, we promised to extend in the same way at each red or blue leaf of T_n and S_n , but we also need that $T_{n+1} - v \cong S_{n+1} - w$. The solution is to copy the same subgraph appearing beyond r in Fig. 1, including its coloured leaves, onto all the other red leaves of S_n and T_n . Similarly we copy the subgraph appearing beyond the blue leaf b of S_n onto all other blue leaves of S_n and

T_n . In doing so, we create new red and blue leaves, and we will keep adding, step by step, further copies of the graphs appearing beyond r and b in Fig. 1 respectively onto all red and blue leaves of everything we have constructed so far.

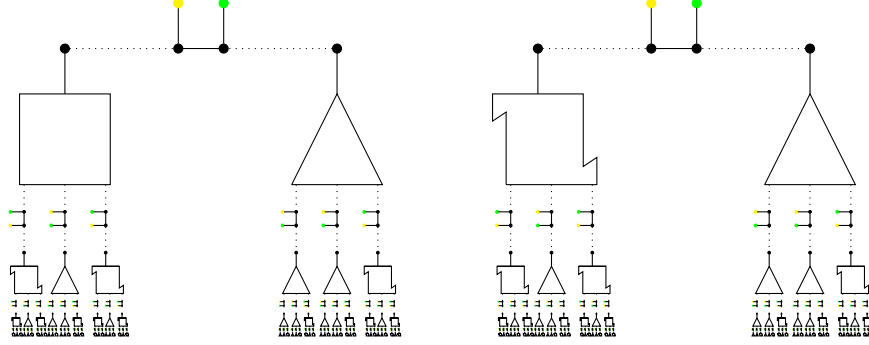


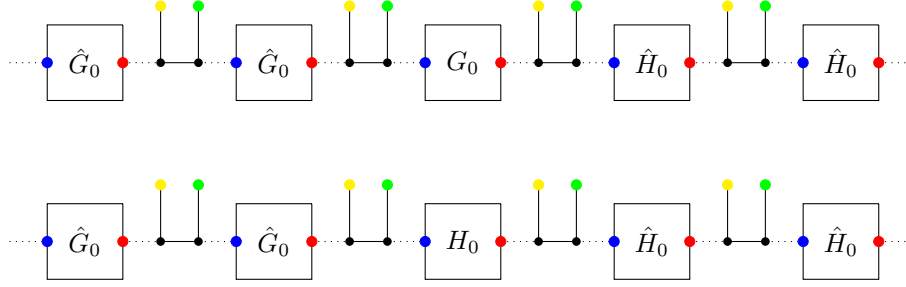
FIGURE 2. A sketch of T_{n+1} and S_{n+1} after countably many steps.

After countably many steps we have dealt with all red and blue leaves, and it can be checked that both our targets are achieved. We take these new trees to be S_{n+1} and T_{n+1} . They are non-isomorphic, as after removing all long non-branching paths, T_{n+1} contains T_n as a component, whereas S_{n+1} does not.

2.2. The one-ended case. To construct a one-ended non-reconstructible graph, we initially follow the same strategy as in the tree case and build locally finite graphs G_n and H_n and some partial hypomorphisms between them. Simultaneously, however, we will also build one-ended locally finite graphs of a grid-like form $F_n \sqcap \mathbb{N}$ (the Cartesian product of a locally finite tree F_n with a ray) which share certain symmetries with G_n and H_n . These will allow us to glue $F_n \sqcap \mathbb{N}$ onto both G_n and H_n , in order to make them one-ended, without spoiling the partial hypomorphisms. Let us illustrate this idea by explicitly describing the first few steps of the construction.

We start with two non-isomorphic graphs G_0 and H_0 , such that G_0 and H_0 each have exactly one red and one blue leaf. After stage one of our algorithm, our approximations to G_1 and H_1 as in Figure 1 contain, in each of G_0 , \hat{H}_0 , \hat{G}_0 and H_0 , one coloured leaf. In stage two, we add copies of these graphs recursively. It follows that the resulting graphs G'_1 and H'_1 have the global structure of a double ray, along which parts corresponding to copies of G_0 , \hat{H}_0 , \hat{G}_0 and H_0 appear in a repeating pattern. Crucially, however, each graph G'_1 and H'_1 has infinitely many yellow and green leaves, which appear in an alternating pattern extending to infinity in both directions along the double ray.

Consider the minor F_1 of G'_1 obtained by collapsing every subgraph corresponding to G_0 , \hat{H}_0 , \hat{G}_0 and H_0 to a single point. Write $\psi_G: G'_1 \rightarrow F_1$ for the quotient map. Then F_1 is a double ray with alternating coloured leaves hanging off it. Note that we could have started with H'_1 and obtained the same F_1 . In other words, F_1 approximates the global structures of both G'_1 and H'_1 . Consider the one-ended grid-like graph $F_1 \sqcap \mathbb{N}$, where we let $F_1 \sqcap \{0\}$ inherit the colours from F_1 . We now form G_1 and H_1 by gluing $F_1 \sqcap \mathbb{N}$ onto G'_1 , by identifying corresponding coloured

FIGURE 3. A sketch of G_1 (above) and H_1 (below).

vertices y and $\psi_G(y)$, and similarly for H_2 .¹ Since the coloured leaves contained both ends of our graphs in their closure, the graphs G_1 and H_1 are now one-ended.

It remains to check that our partial isomorphism $h_1: G'_1 - v_1 \rightarrow H'_1 - w_1$ guaranteed by step two can be extended to $G_1 - v_1 \rightarrow H_1 - w_1$. This can be done essentially because of the following property: let us write $\mathcal{L}(\cdot)$ for the set of coloured leaves. It can be checked that there is an automorphism $\pi_1: F_1 \rightarrow F_1$ such that the diagram

$$\begin{array}{ccc} \mathcal{L}(G'_1) & \xrightarrow{h_1 \upharpoonright \mathcal{L}(G_1)} & \mathcal{L}(H'_1) \\ \psi_G \downarrow & & \downarrow \psi_H \\ \mathcal{L}(F_1) & \xrightarrow{\pi_1 \upharpoonright \mathcal{L}(F_1)} & \mathcal{L}(F_1) \end{array}$$

is colour-preserving and commutes. Hence, $\pi_1 \circ \text{id}$ is an automorphism of $F_1 \square \mathbb{N}$ which is compatible with our gluing procedure, so it can be combined with h_1 to give us the desired isomorphism.

We are now ready to describe the general step. Instead of describing F_n as a minor of G_n , which no longer works naively at later steps, we will directly build F_n by recursion, so that it satisfies the properties of the above diagram.

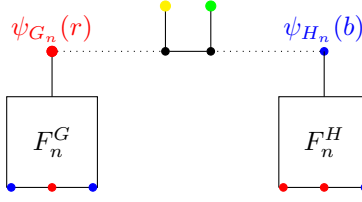
Suppose at step n we have constructed locally finite graphs G_n and H_n , and also a locally finite tree F_n where some leaves are coloured in one of two colours. Furthermore, suppose we have a family of isomorphisms

$$\mathcal{H}_n = \{h_x: G_n - x \rightarrow H_n - \varphi(x): x \in X_n\},$$

for some subset $X_n \subset V(G_n)$, a family of isomorphisms $\Pi_n = \{\pi_x: F_n \rightarrow F_n: x \in X_n\}$, and colour-preserving bijections $\psi_{G_n}: \mathcal{L}(G_n) \rightarrow \mathcal{L}(F_n)$ and $\psi_{H_n}: \mathcal{L}(H_n) \rightarrow \mathcal{L}(F_n)$ such that the corresponding commutative diagram from above holds for each x . We construct G'_{n+1} and H'_{n+1} according to stages one and two of the previous algorithm. As before our isomorphisms h_x will lift to isomorphisms between $G'_{n+1} - x$ and $H'_{n+1} - \varphi(x)$.

Algorithm Stage Three. As indicated in Figure 4, we take two copies F_n^G and F_n^H of F_n , and glue them together mimicking stage one of the algorithm, i.e. connect $\psi_{G_n}(r)$ in F_n^G by a path of length three to $\psi_{H_n}(b)$ in F_n^H , and attach two new leaves coloured yellow and green in the middle of the path. Call the resulting graph \tilde{F}_n .

¹For technical reasons, in the actual construction we identify $\psi_G(y)$ with the corresponding base vertex of the leaf y in G'_1 . In this way the coloured leaves of G'_1 remain leaves, and we can continue our recursive construction.

FIGURE 4. The auxiliary graph \tilde{F}_n .

We then apply stage two of the algorithm to this graph, gluing again and again onto every blue vertex a copy of the graph of \tilde{F}_n behind $\psi_{H_n}(b)$, and similarly for every red leaf, to obtain a tree F_{n+1} . Since this procedure is, in structural terms, so similar to the construction of G'_{n+1} and H'_{n+1} , it can be shown that we do obtain a colour-preserving commuting diagram of the form

$$\begin{array}{ccc}
 \mathcal{L}(G'_{n+1}) & \xrightarrow{h_x \upharpoonright \mathcal{L}(G'_{n+1})} & \mathcal{L}(H'_{n+1}) \\
 \psi_{G_{n+1}} \downarrow & & \downarrow \psi_{H_{n+1}} \\
 \mathcal{L}(F_{n+1}) & \xrightarrow{\pi_x \upharpoonright \mathcal{L}(F_{n+1})} & \mathcal{L}(F_{n+1})
 \end{array}$$

As before, this means that we can indeed glue together G'_{n+1} and $F_{n+1} \square \mathbb{N}$, and H'_{n+1} and $F_{n+1} \square \mathbb{N}$ to obtain one-ended graphs G_{n+1} and H_{n+1} as desired.

At the end of our construction, after countably many steps, we have built two graphs G and H which are hypomorphic, and for the same reasons as in the tree case the two graphs will not be isomorphic. Further, since all G_n and H_n are one-ended, so will be G and H .

2.3. The countably-ended case. In order to produce hypomorphic graphs with countably many ends we follow the same procedure as for the one-ended case, except that we start with one-ended (non-isomorphic) graphs G_0 and H_0 .

After the first and second stage of our algorithm, the resulting graphs G'_1 and H'_1 will again consist of infinitely many copies of G_0 and H_0 glued together along a double ray. After gluing $F_1 \square \mathbb{N}$ to these graphs as before, we obtain graphs with one thick end, with many coloured leaves tending to that end, as well as infinitely many thin ends, coming from the copies of G_0 and H_0 , each of which contained a ray. These thin ends will eventually be rays, and so have no coloured leaves tending towards them. This guarantees that in the next step, when we glue $F_2 \square \mathbb{N}$ onto G'_2 and H'_2 , the thin ends will not be affected, and that all the other ends in the graph will be amalgamated into one thick end.

Then, in each stage of the construction, the graphs G_n and H_n will have exactly one thick end, again with many coloured leaves tending towards it, and infinitely many thin ends each of which is eventually a ray. This property lifts to the graphs G and H constructed in the limit: they will have one thick end and infinitely many ends which are eventually rays. However, since G and H are countable, there can only be countably many of these rays. Hence the two graphs G and H have countably many ends in total, and as before they will be hypomorphic but not isomorphic.

3. CLOSURE WITH RESPECT TO PROMISES

A *bridge* in a graph G is an edge $e = \{x, y\}$ such that x and y lie in different components of $G - e$. Given a directed bridge $\vec{e} = \vec{x}y$ in some graph $G = (V, E)$, we denote by $G(\vec{e})$ the unique component of $G - e$ containing the vertex y . We think of $G(\vec{e})$ as a rooted graph with root y .

Definition 3.1 (Promise structure). *A promise structure $\mathcal{P} = (G, \vec{P}, \mathcal{L})$ is a triple consisting of:*

- a graph G ,
- $\vec{P} = \{\vec{p}_i : i \in I\}$ a set of directed bridges $\vec{P} \subset \vec{E}(G)$, and
- $\mathcal{L} = \{L_i : i \in I\}$ a set of pairwise disjoint sets of leaves of G .

We insist further that, if the component $G(\vec{p}_i)$ consists of a single leaf $c \in L_j$, then $i = j$.

Often, when the context is clear, we will not make a distinction between \mathcal{L} and the set $\bigcup_i L_i$, for notational convenience.

We call an edge $\vec{p}_i \in \vec{P}$ a *promise edge*, and leaves $\ell \in L_i$ *promise leaves*. A promise edge $\vec{p}_i \in \vec{P}$ is called a *placeholder-promise* if the component $G(\vec{p}_i)$ consists of a single leaf $c \in L_i$, which we call a *placeholder-leaf*. We write

$$\mathcal{L}_p = \{L_i : \vec{p}_i \text{ a placeholder-promise}\} \text{ and } \mathcal{L}_q = \mathcal{L} \setminus \mathcal{L}_p.$$

Given a leaf ℓ in G , there is a unique edge $q_\ell \in E(G)$ incident with ℓ , and this edge has a natural orientation \vec{q}_ℓ towards ℓ . Informally, we think of $\ell \in L_i$ as the ‘promise’ that if we extend G to a graph $H \supset G$, we will do so in such a way that $H(\vec{q}_\ell) \cong H(\vec{p}_i)$.

Definition 3.2 (Leaf extension). *Given an inclusion $H \supseteq G$ of graphs and a set L of leaves of G , H is called a leaf extension, or more specifically an L -extension, of G , if:*

- every component of H contains precisely one component of G , and
- every component of $H - G$ is adjacent to a unique vertex l of G , and we have $l \in L$.

In [4], given a promise structure $\mathcal{P} = (G, \vec{P}, \mathcal{L})$, it is shown how to construct a graph $\text{cl}(G) \supset G$ which has the following properties.

Proposition 3.3 (Closure w.r.t a promise structure, cf. [4, Proposition 3.3]). *Let G be a graph and let $(G, \vec{P}, \mathcal{L})$ be a promise structure. Then there is a graph $\text{cl}(G)$, called the closure of G with respect to \mathcal{P} , such that:*

(cl.1) $\text{cl}(G)$ is an \mathcal{L}_q -extension of G ,

(cl.2) for every $\vec{p}_i \in \vec{P}$ and all $\ell \in L_i$,

$$\text{cl}(G)(\vec{p}_i) \cong \text{cl}(G)(\vec{q}_\ell)$$

are isomorphic as rooted graphs.

Since the existence of $\text{cl}(G)$ is crucial to our proof, we briefly remind the reader how to construct such a graph. As a first approximation, in order to try to achieve ((cl.2)), we glue a copy of the component $G(\vec{p}_i)$ onto each leaf $\ell \in L_i$, for each $i \in I$. We call this the *1-step extension* $G^{(1)}$ of G . If there were no promise leaves in the

component $G(\vec{p}_i)$, then the promises in L_i would be satisfied. However, if there are, then we have grown $G(\vec{p}_i)$ by adding copies of various $G(\vec{p}_j)$ s behind promise leaves appearing in $G(\vec{p}_i)$.

However, remembering all promise leaves inside the newly added copies of $G(\vec{p}_i)$ we glued behind each $\ell \in L_i$, we continue this process indefinitely, growing the graph one step at a time by gluing copies of (the original) $G(\vec{p}_i)$ to promise leaves ℓ' which have appeared most recently as copies of $\ell \in L_i$. After a countable number of steps the resulting graph $\text{cl}(G)$ satisfies Proposition 3.3. We note also that the maximum degree of $\text{cl}(G)$ equals that of G .

Definition 3.4 (Promise-respecting map). *Let G be a graph, $\mathcal{P} = (G, \vec{P}, \mathcal{L})$ be a promise structure on G , and let T_1 and T_2 be two components of G .*

Given $x \in T_1$ and $y \in T_2$, a bijection $\varphi: T_1 - x \rightarrow T_2 - y$ is \vec{P} -respecting (with respect to \mathcal{P}) if the image of $L_i \cap T_1$ under φ is $L_i \cap T_2$ for all i .

We can think of \mathcal{P} as defining a $|\vec{P}|$ -colouring on some sets of leaves. Then a mapping is \vec{P} -respecting if it preserves leaf colours.

Suppose that \vec{p}_i is a placeholder promise, and $G = H^{(0)} \subseteq H^{(1)} \subseteq \dots$ is the sequence of 1-step extensions whose direct limit is $\text{cl}(G)$. Then, if we denote by $L_i^{(n)}$ the set of promise leaves associated with \vec{p}_i in $H^{(n)}$, it follows that $L_i^{(n)} \supseteq L_i^{(n-1)}$ since $G(\vec{p}_i)$ is just a single vertex $c_i \in L_i$. For every placeholder promise $\vec{p}_i \in \vec{P}$, we define $\text{cl}(L_i) = \bigcup_n L_i^{(n)}$.

Definition 3.5 (Closure of a promise structure). *The closure of the promise structure $(G, \vec{P}, \mathcal{L})$ is the promise structure $\text{cl}(\mathcal{P}) = (\text{cl}(G), \text{cl}(\vec{P}), \text{cl}(\mathcal{L}))$, where:*

- $\text{cl}(\vec{P}) = \{\vec{p}_i : \vec{p}_i \in \vec{P} \text{ is a placeholder-promise}\}$,
- $\text{cl}(\mathcal{L}) = \{\text{cl}(L_i) : \vec{p}_i \in \vec{P} \text{ is a placeholder-promise}\}$.

Proposition 3.6 ([4, Proposition 3.3]). *Let G be a graph and let $(G, \vec{P}, \mathcal{L})$ be a promise structure. Then $\text{cl}(G)$ satisfies:*

(cl.3) *for every $\vec{p}_i \in \vec{P}$ and every $\ell \in L_i$,*

$$\text{cl}(G)(\vec{p}_i) \cong \text{cl}(G)(\vec{q}_\ell)$$

are isomorphic as rooted graphs, and this isomorphism is $\text{cl}(\vec{P})$ -respecting with respect to $\text{cl}(\mathcal{P})$.

It is precisely this property (cl.3) of the promise closure that will allow us to maintain partial hypomorphisms during our recursive construction.

The last two results of this section serve as preparation for growing G_{n+1} , H_{n+1} and F_{n+1} ‘in parallel’, as outlined in the third stage of the algorithm in §2.2. If $\mathcal{L} = \{L_i : i \in I\}$ and $\mathcal{L}' = \{L'_i : i \in I\}$, we say a map $\psi : \bigcup \mathcal{L} \rightarrow \bigcup \mathcal{L}'$ is *colour-preserving* if $\psi(L_i) \subseteq L'_i$ for every i .

Lemma 3.7. *Let $(G, \vec{P}, \mathcal{L})$ and $(G', \vec{P}', \mathcal{L}')$ be promise structures, and let $G = H^{(0)} \subseteq H^{(1)} \subseteq \dots$ and $G' = H'^{(0)} \subseteq H'^{(1)} \subseteq \dots$ be 1-step extensions approximating their respective closures.*

Assume that $\vec{P} = \{\vec{p}_1, \dots, \vec{p}_k\}$ and $\vec{P}' = \{\vec{r}_1, \dots, \vec{r}_k\}$, and that there is a colour-preserving bijection

$$\psi: \bigcup \mathcal{L} \rightarrow \bigcup \mathcal{L}'$$

such that (recall that $\mathcal{L}(\cdot)$ is the set of leaves of a graph that are in \mathcal{L})

$$\psi \upharpoonright G(\vec{p}_i): \mathcal{L}(G(\vec{p}_i)) \rightarrow \mathcal{L}'(G'(\vec{r}_i))$$

is still a colour-preserving bijection for all $\vec{p}_i \in \vec{P}$.

Then for each $i \leq k$ there is a sequence of colour-preserving bijections

$$\alpha_n^i: \mathcal{L}(H^{(n)}(\vec{p}_i)) \rightarrow \mathcal{L}'(H'^{(n)}(\vec{r}_i))$$

such that α_{n+1}^i extends α_n^i .

Proof. Fix i . We proceed by induction on n . Put $\alpha_0^i := \psi \upharpoonright G(\vec{p}_i)$.

Now suppose that α_n^i exists. To form $H^{(n+1)}(\vec{p}_i)$, we glued a copy of $G(\vec{p}_j)$ to each $\ell \in L_j^{(n)} \cap H^{(n)}(\vec{p}_i)$ for all $j \leq k$, and to construct $H'^{(n+1)}(\vec{r}_i)$, we glued a copy of $G'(\vec{r}_j)$ to each $\ell' \in L_j'^{(n)} \cap H'^{(n)}(\vec{r}_i)$ for all $j \leq k$, in both cases keeping all copies of promise leaves.

By assumption, the second part can be phrased equivalently as: we glued on a copy of $G'(\vec{r}_j)$ to each $\alpha_n^i(\ell)$ for $\ell \in L_j^{(n)} \cap H^{(n)}(\vec{p}_i)$. Thus, we can now combine the bijections $\alpha_n^i(\ell)$ with all the individual bijections ψ between all newly added $G(\vec{p}_j)$ and $G'(\vec{r}_j)$ to obtain a bijection α_{n+1}^i as desired. \square

Corollary 3.8. *In the above situation, for each i there is a colour-preserving bijection α^i between $\mathcal{L}(\text{cl}(G)(\vec{p}_i))$ and $\mathcal{L}'(\text{cl}(G')(\vec{r}_i))$ with respect to the promise closures $\text{cl}(\mathcal{P})$ and $\text{cl}(\mathcal{P}')$.*

Proof. Put $\alpha^i = \bigcup_n \alpha_n^i$. Because all α_n^i respected all colours, they respect in particular the placeholder promises which make up $\text{cl}(\mathcal{P})$ and $\text{cl}(\mathcal{P}')$. \square

4. THICKENING THE GRAPH

In this section, we lay the groundwork for the third stage of our algorithm, as outlined in §2.2. Our aim is to clarify how gluing a one-ended graph F onto a graph G affects automorphisms and the end-space of the resulting graph.

Definition 4.1 (Gluing sum). *Given two graphs G and F , and a bijection ψ with $\text{dom}(\psi) \subseteq V(G)$ and $\text{ran}(\psi) \subseteq V(F)$, the gluing sum of G and F along ψ , denoted by $G \oplus_\psi F$, is the quotient graph $(G \cup F) / \sim$ where $v \sim \psi(v)$ for all $v \in \text{dom}(\psi)$.*

Our first lemma of this section explains how a partial isomorphism from $G_n - x$ to $H_n - \phi(x)$ in our construction can be lifted to the gluing sum of G_n and H_n with a graph F respectively.

Lemma 4.2. *Let G , H and F be graphs, and consider two gluing sums $G \oplus_{\psi_G} F$ and $H \oplus_{\psi_H} F$ along partial bijections ψ_G and ψ_H . Suppose there exists an isomorphism $h: G - x \rightarrow H - y$ that restricts to a bijection between $\text{dom}(\psi_G)$ and $\text{dom}(\psi_H)$.*

Then h extends to an isomorphism $(G \oplus_{\psi_G} F) - x \rightarrow (H \oplus_{\psi_H} F) - y$ provided there is an automorphism π of F such that $\pi \circ \psi_G(v) = \psi_H \circ h(v)$ for all $v \in \text{dom}(\psi_G)$.

Proof. We verify that the map

$$\hat{h}: (G \oplus_{\psi_G} F) - x \rightarrow (H \oplus_{\psi_H} F) - y, \quad v \mapsto \begin{cases} h(v) & \text{if } v \in G - x, \text{ and} \\ \pi(v) & \text{if } v \in F \end{cases}$$

is a well-defined isomorphism. It is well-defined, since if $v \sim \psi_G(v)$ in $G \oplus_{\psi_G} F$, then $\hat{h}(v) \sim \hat{h}(\psi_G(v))$ in $H \oplus_{\psi_H} F$ by assumption on π . Moreover, since h and π are isomorphisms, it follows that \hat{h} is an isomorphism, too. \square

For the remainder of this section, all graphs are assumed to be locally finite. A *ray* in a graph G is a one-way infinite path. Given a ray R , then for any finite vertex set $S \subset V(G)$ there is a unique component $C(R, S)$ of $G - S$ containing a tail of R . An *end* in a graph is an equivalence class of rays under the relation

$$R \sim R' \Leftrightarrow \text{for every finite vertex set } S \subset V(G) \text{ we have } C(R, S) = C(R', S).$$

We denote by $\Omega(G)$ the set of ends in the graph G , and write $C(\omega, S) := C(R, S)$ with $R \in \omega$. Let $\Omega(\omega, S) = \{\omega' : C(\omega', S) = C(\omega, S)\}$. The singletons $\{v\}$ for $v \in V(G)$ and sets of the form $C(\omega, S) \cup \Omega(\omega, S)$ generate a compact metrizable topology on the set $V(G) \cup \Omega(G)$, which is known in the literature as $|G|$.² This topology allows us to talk about the closure of a set of vertices $X \subset V(G)$, denoted by \overline{X} . Write $\partial(X) = \overline{X} \setminus X = \overline{X} \cap \Omega(G)$ for the boundary of X : the collection of all ends in the closure of X . Then an end $\omega \in \Omega(G)$ lies in $\partial(X)$ if and only if for every finite vertex set $S \subset V(G)$, we have $|X \cap C(\omega, S)| = \infty$. Therefore $\Omega(G) = \partial(X)$ if and only if for every finite vertex set $S \subset V(G)$, every infinite component of $G - S$ meets X infinitely often. In this case we say that X is *dense* for $\Omega(G)$.

Finally, an end $\omega \in \Omega(G)$ is *free* if for some S , the set $\Omega(\omega, S) = \{\omega\}$. Then $\Omega'(G)$ denotes the *non-free* (or *limit-*)ends. Note that $\Omega'(G)$ is a closed subset of $\Omega(G)$.

Lemma 4.3. *For locally finite connected graphs G and F , consider the gluing sum $G \oplus_{\psi} F$ for a partial bijection ψ . If F is one-ended and $\text{dom}(\psi)$ is infinite, then $\Omega(G \oplus_{\psi} F) \cong \Omega(G) / \partial(\text{dom}(\psi))$.*

Proof. Note first that for locally finite graphs G and F , also $G \oplus_{\psi} F$ is locally finite. Observe further that all rays of the unique end of F are still equivalent in $G \oplus_{\psi} F$, and so $G \oplus_{\psi} F$ has an end $\hat{\omega}$ containing the single end of F .

We are going to define a continuous surjection $f: \Omega(G) \rightarrow \Omega(G \oplus_{\psi} F)$ with the property that f has precisely one non-trivial fibre, namely $f^{-1}(\hat{\omega}) = \partial(\text{dom}(\psi))$. It then follows from definition of the quotient topology that f induces a continuous bijection from the compact space $\Omega(G) / \partial(\text{dom}(\psi))$ to the Hausdorff space $\Omega(G \oplus_{\psi} F)$, which, as such, is necessarily a homeomorphism.

The mapping f is defined as follows. Given an end $\omega \in \Omega(G) \setminus \partial(\text{dom}(\psi))$, there is a finite $S \subset V(G)$ such that $C(\omega, S) \cap \text{dom}(\psi) = \emptyset$, and so $C = C(\omega, S)$ is also a component of $(G \oplus_{\psi} F) - S$, which is disjoint from F . Define f to be the identity between $\Omega(G) \cap \overline{C}$ and $\Omega(G \oplus_{\psi} F) \cap \overline{C}$, while for all remaining ends $\omega \in \Omega(G) \cap \text{dom}(\psi)$, we put $f(\omega) = \hat{\omega}$.

To see that this assignment is continuous at $\omega \in \Omega(G) \cap \overline{\text{dom}(\psi)}$, it suffices to show that $C := C(\omega, S) \subset G - S$ is a subset of $C' := C(\hat{\omega}, S) \subset (G \oplus_{\psi} F) - S$

²Normally $|G|$ is defined on the 1-complex of G together with its ends, but for our purposes it will be enough to just consider the subspace $V(G) \cup \Omega(G)$. See the survey paper of Diestel [6] for further details.

for any finite set $S \subset G \oplus_\psi F$. To see this inclusion, note that by choice of ω , we have $|\text{dom}(\psi) \cap C| = \infty$. At the same time, since F is both one-ended and locally finite, $F - S$ has precisely one infinite component D and $F - D$ is finite, so as ψ is a bijection, there is $v \in \text{dom}(\psi) \cap C$ with $\psi(v) \in D$ (in fact, there are infinitely many such v). Since v and $\psi(v)$ get identified in $G \oplus_\psi F$, we conclude that $C \cup D$ is connected in $(G \oplus_\psi F) - S$, and hence that $C \cup D \subset C'$ as desired.

Finally, to see that \overline{f} is indeed surjective, note first that the fact that $\text{dom}(\psi)$ is infinite implies that $\overline{\text{dom}(\psi)} \cap \Omega(G) \neq \emptyset$, and so $\hat{\omega} \in \text{ran}(f)$. Next, consider an end $\omega \in \Omega(G \oplus_\psi F)$ different from $\hat{\omega}$. Find a finite separator $S \subset V(G \oplus_\psi F)$ such that $C(\omega, S) \neq C(\hat{\omega}, S)$. It follows that $\text{dom}(\psi) \cap C(\omega, S)$ is finite. So there is a finite $S' \supseteq S$ such that $C := C(\omega, S') \neq C(\hat{\omega}, S')$ and $\text{dom}(\psi) \cap C = \emptyset$. So by definition, f is a bijection between $\Omega(G) \cap \overline{C}$ and $\Omega(G \oplus_\psi F) \cap \overline{C}$, so $\omega \in \text{ran}(f)$. \square

Corollary 4.4. *Under the above assumptions, if $\text{dom}(\psi)$ is dense for $\Omega(G)$, then $G \oplus_\psi F$ is one-ended.*

Corollary 4.5. *Under the above assumptions, if $\overline{\text{dom}(\psi)} \cap \Omega(G) = \Omega'(G)$, then $G \oplus_\psi F$ has at most one non-free end.*

We remark that more direct proofs for Corollaries 4.4 and 4.5 can be given that do not need the full power of Lemma 4.3.

5. THE CONSTRUCTION

5.1. Preliminary definitions. In the precise statement of our construction in §5.2, we are going to employ the following notation.

Definition 5.1 (Mii-path). *A path $P = v_0, v_1, \dots, v_n$ in a graph G is called internally isolated if $\deg_G(v_i) = 2$ for all internal vertices v_i for $0 < i < n$. The path P is maximal internally isolated (or mii for short) if in addition $\deg_G(v_0) \neq 2 \neq \deg_G(v_n)$. An infinite path $P = v_0, v_1, v_2, \dots$ is mii if $\deg_G(v_0) \neq 2$ and $\deg_G(v_i) = 2$ for all $i \geq 1$.*

Definition 5.2 (Mii-spectrum). *The mii-spectrum of G is*

$$\Sigma(G) := \{k \in \mathbb{N} : G \text{ contains an mii-path of length } k\}.$$

If $\Sigma(G)$ is finite, we let $\sigma_0(G) = \max \Sigma(G)$ and $\sigma_1(G) = \max (\Sigma(G) \setminus \{\sigma_0(G)\})$.

Lemma 5.3. *Let e be an edge of a locally finite graph G . If $\Sigma(G)$ is finite, then $\Sigma(G - e)$ is finite.*

Proof. Observe first that every vertex of degree ≤ 2 in any graph can lie on at most one mii-path.

We now claim that for an edge $e = xy$, there are at most two finite mii-paths in $G - e$ which are not subpaths of finite mii-paths of G .

Indeed, if $\deg x = 3$ in G , then x can now be the interior vertex of one new finite mii-path in $G - e$. And if $\deg x = 2$ in G , then x can now be end-vertex of one new finite mii-path in $G - e$ (this is relevant if x lies on an infinite mii-path of G). The argument is for y is the same, so the claim follows. \square

Definition 5.4 (Spectrally distinguishable). *Given two graphs G and H , we say that G and H are spectrally distinguishable if there is some $k \geq 3$ such that $k \in \Sigma(G) \Delta \Sigma(H) = \Sigma(G) \setminus \Sigma(H) \cup \Sigma(H) \setminus \Sigma(G)$.*

Note that being spectrally distinguishable is a strong certificate for being non-isomorphic.

Definition 5.5 (*k*-ball). *For G a subgraph of H , and $k > 0$, the k -ball $\text{Ball}_H(G, k)$ is the induced subgraph of H on the set of vertices at distance at most k of some vertex of G .*

Definition 5.6 (proper Mii-extension; infinite growth). *Let G be a graph, B a subset of leaves of G , and H a component of G .*

- *A graph $\hat{G} \supset H$ is an mii-extension of H at B to length k if $\text{Ball}_{\hat{G}}(H, k)$ can be obtained from H by adjoining, at each vertex $l \in B \cap V(H)$, a new path of length k starting at l , and a new leaf whose only neighbour is l .³*
- *A leaf l in a graph G is proper if the unique neighbour of l in G has degree ≥ 3 . An mii-extension is called proper if every leaf in B is proper.*
- *An mii-extension \hat{G} of G is of infinite growth if every component of $\hat{G} - G$ is infinite.*

5.2. The back-and-forth construction. Our aim in this section is to prove our main theorem announced in the introduction.

Theorem 1.2. *There are two hypomorphic connected one-ended infinite graphs G and H with maximum degree five such that G is not isomorphic to H .*

To do this we shall recursively construct, for each $n \in \mathbb{N}$,

- disjoint rooted connected graphs G_n and H_n ,
- disjoint sets R_n and B_n of proper leaves of the graph $G_n \cup H_n$,
- trees F_n ,
- disjoint sets R'_n and B'_n of leaves of F_n ,
- bijections $\psi_{G_n} : V(G_n) \cap (R_n \cup B_n) \rightarrow R'_n \cup B'_n$ and $\psi_{H_n} : V(H_n) \cap (R_n \cup B_n) \rightarrow R'_n \cup B'_n$,
- finite sets $X_n \subset V(G_n)$ and $Y_n \subset V(H_n)$, and bijections $\varphi_n : X_n \rightarrow Y_n$,
- a family of isomorphisms $\mathcal{H}_n = \{h_{n,x} : G_n - x \rightarrow H_n - \varphi_n(x) : x \in X_n\}$,
- a family of automorphisms $\Pi_n = \{\pi_{n,x} : F_n \rightarrow F_n : x \in X_n\}$,
- a strictly increasing sequence of integers $k_n \geq 2$,

such that for all $n \in \mathbb{N}$:⁴

- (†1) $G_{n-1} \subset G_n$ and $H_{n-1} \subset H_n$ as induced subgraphs,
- (†2) the vertices of G_n and H_n all have degree at most 5,
- (†3) the vertices of F_n all have degree at most 3,
- (†4) the root of G_n is in R_n and the root of H_n is in B_n ,
- (†5) $\sigma_0(G_n) = \sigma_0(H_n) = k_n$,
- (†6) G_n and H_n are spectrally distinguishable,
- (†7) G_n and H_n have at most one end,
- (†8) $\Omega(G_n \cup H_n) \subset \overline{R_n \cup B_n}$,
- (†9) (a) G_n is a (proper) mii-extension of infinite growth of G_{n-1} at $R_{n-1} \cup B_{n-1}$ to length $k_{n-1} + 1$, and
(b) $\text{Ball}_{G_n}(G_{n-1}, k_{n-1} + 1)$ does not meet $R_n \cup B_n$,
- (†10) (a) H_n is a (proper) mii-extension of infinite growth of H_{n-1} at $R_{n-1} \cup B_{n-1}$ to length $k_{n-1} + 1$, and

³We note that this is a slightly different definition of an mii-extension to that in [4].

⁴If the statement involves an object indexed by $n - 1$ we only require that it holds for $n \geq 1$.

- (b) $\text{Ball}_{H_n}(H_{n-1}, k_{n-1} + 1)$ does not meet $R_n \cup B_n$,
- (†11) there are enumerations $V(G_n) = \{t_j : j \in J_n\}$ and $V(H_n) = \{s_j : j \in J_n\}$ such that
- $J_{n-1} \subset J_n \subset \mathbb{N}$,
 - $\{t_j : j \in J_n\}$ extends the enumeration $\{t_j : j \in J_{n-1}\}$ of $V(G_{n-1})$, and similarly for $\{s_j : j \in J_n\}$,
 - $|\mathbb{N} \setminus J_n| = \infty$,
 - $\{0, 1, \dots, n\} \subset J_n$,
- (†12) $\{t_j, s_j : j \leq n\} \cap (R_n \cup B_n) = \emptyset$,
- (†13) the finite sets of vertices X_n and Y_n satisfy $|X_n| = n = |Y_n|$, and
- $X_{n-1} \subset X_n$ and $Y_{n-1} \subset Y_n$,
 - $\varphi_n \upharpoonright X_{n-1} = \varphi_{n-1}$,
 - $\{t_j : j \leq \lfloor (n-1)/2 \rfloor\} \subset X_n$ and $\{s_j : j \leq \lfloor n/2 \rfloor - 1\} \subset Y_n$,
 - $(X_n \cup Y_n) \cap (R_n \cup B_n) = \emptyset$,
- (†14) the families of isomorphisms \mathcal{H}_n satisfy
- $h_{n,x} \upharpoonright (G_{n-1} - x) = h_{n-1,x}$ for all $x \in X_{n-1}$,
 - the image of $R_n \cap V(G_n)$ under $h_{n,x}$ is $R_n \cap V(H_n)$,
 - the image of $B_n \cap V(G_n)$ under $h_{n,x}$ is $B_n \cap V(H_n)$ for all $x \in X_n$.
- (†15) the families of automorphisms Π_n satisfy
- $\pi_{n,x} \upharpoonright R'_n$ is a permutation of R'_n for each $x \in X_n$,
 - $\pi_{n,x} \upharpoonright B'_n$ is a permutation of B'_n for each $x \in X_n$,
 - for each $x \in X_n$, the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{L}(G_n) & \xrightarrow{h_{n,x} \upharpoonright \mathcal{L}(G_n)} & \mathcal{L}(H_n) \\
 \psi_{G_n} \downarrow & & \downarrow \psi_{H_n} \\
 \mathcal{L}(F_n) & \xrightarrow{\pi_{n,x} \upharpoonright \mathcal{L}(F_n)} & \mathcal{L}(F_n)
 \end{array}$$

I.e. for every $\ell \in \mathcal{L}(G_n) := V(G_n) \cap (R_n \cup B_n)$ we have $\pi_{n,x}(\psi_{G_n}(\ell)) = \psi_{H_n}(h_{n,x}(\ell))$.

5.3. The construction yields the desired non-reconstructible one-ended graphs. By property (†1), we have $G_0 \subset G_1 \subset G_2 \subset \dots$ and $H_0 \subset H_1 \subset H_2 \subset \dots$. Let G and H be the union of the respective sequences. Then both G and H are connected, and as a consequence of (†2), both graphs have maximum degree 5.

We claim that the map $\varphi = \bigcup_n \varphi_n$ is a hypomorphism between G and H . Indeed, it follows from (†11) and (†13) that φ is a well-defined bijection from $V(G)$ to $V(H)$. To see that φ is a hypomorphism, consider any vertex x of G . This vertex appears as some t_j in our enumeration of $V(G)$, so the map

$$h_x = \bigcup_{n > 2j} h_{n,x} : G - x \rightarrow H - \varphi(x),$$

is a well-defined isomorphism by (†14) between $G - x$ and $H - \varphi(x)$.

Now suppose for a contradiction that there exists an isomorphism $f : G \rightarrow H$. Then $f(t_0)$ is mapped into H_n for some $n \in \mathbb{N}$. Properties (†5) and (†9) imply that after deleting all mii-paths in G of length $> k_n$, the connected component C of t_0 is a leaf extension of G_n adding one further leaf to every vertex in $V(G_n) \cap (R_n \cup B_n)$. Similarly, properties (†5) and (†10) imply that after deleting all mii-paths in H of length $> k_n$, the connected component D of $f(t_0)$ is a leaf-extension of H_n adding

one further leaf to every vertex in $V(H_n) \cap (R_n \cup B_n)$. Note that f restricts to an isomorphism between C and D . However, since C and D are proper extensions, we have $\Sigma(C) \Delta \Sigma(G_n) \subseteq \{1, 2\}$ and $\Sigma(D) \Delta \Sigma(H_n) \subseteq \{1, 2\}$. Hence, since G_n and H_n are spectrally distinguishable by $(\dagger 6)$, so are C and D , a contradiction. We have established that G and H are non-isomorphic reconstructions of each other.

Finally, for G being one-ended, we now show that for every finite vertex separator $S \subset V(G)$, the graph $G - S$ has only one infinite component (the argument for H is similar). Suppose for a contradiction $G - S$ has two infinite components C_1 and C_2 . Consider n large enough such that $S \subset V(G_n)$. Since G_k is one-ended for all k by $(\dagger 7)$, we may assume that $C_1 \cap G_k$ falls apart into finite components for all $k \geq n$. Since C_1 is infinite and connected, it follows from $(\dagger 9)(b)$ that C_1 intersects $G_{n+1} - G_n$. But since G_{n+1} is an mii-extension of G_n of infinite growth by $(\dagger 9)(a)$, we see that that $C_1 \cap (G_{n+1} - G_n)$ contains an infinite component, a contradiction.

5.4. The base case: there are finite rooted graphs G_0 and H_0 satisfying requirements $(\dagger 1)$ – $(\dagger 15)$. Choose a pair of spectrally distinguishable, equally sized graphs G_0 and H_0 with maximum degree ≤ 5 and $\sigma_0(G_0) = \sigma_0(H_0) = k_0$. Pick a proper leaf each as roots $r(G_0)$ and $r(H_0)$ for G_0 and H_0 , and further proper leaves $\ell_b \in G_0$ and $\ell_r \in H_0$.

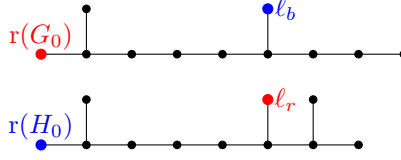


FIGURE 5. A possible choice for the finite rooted graphs G_0 and H_0 .

Define $R_0 = \{r(G_0), \ell_r\}$ and $B_0 = \{r(H_0), \ell_b\}$. We take F_0 to be two vertices x and y joined by an edge, with $R'_0 = \{x\}$ and $B'_0 = \{y\}$ and take ψ_{G_0} to be the unique bijection sending $R_0 \cap G_0$ to R'_0 and $B_0 \cap G_0$ to B'_0 , and similarly for ψ_{H_0} .



FIGURE 6. F_0 .

Let $J_0 = \{0, 1, \dots, |G_0| - 1\}$ and choose enumerations $V(G_0) = \{t_j : j \in J_0\}$ and $V(H_0) = \{s_j : j \in J_0\}$ with $t_0 \neq r(G_0)$ and $s_0 \neq r(H_0)$. Finally we let $X_0 = Y_0 = \mathcal{H}_0 = \emptyset$. It is a simple check that conditions $(\dagger 1)$ – $(\dagger 15)$ are satisfied.

5.5. The inductive step: set-up. Now, assume that we have constructed graphs G_k and H_k for all $k \leq n$ such that $(\dagger 1)$ – $(\dagger 15)$ are satisfied up to n . If $n = 2m$ is even, then we have $\{t_j : j \leq m - 1\} \subset X_n$ and in order to satisfy $(\dagger 13)$ we have to construct G_{n+1} and H_{n+1} such that the vertex t_m is taken care of in our partial hypomorphism. Similarly, if $n = 2m + 1$ is odd, then we have $\{s_j : j \leq m - 1\} \subset Y_n$ and we have to construct G_{n+1} and H_{n+1} such that the vertex s_m is taken care of in our partial hypomorphism. Both cases are symmetric, so let us assume in the following that $n = 2m$ is even.

Now let v be the vertex with the least index in the set $\{t_j : j \in J_n\} \setminus X_n$, i.e.

$$v = t_i \text{ for } i = \min \{j : t_j \in V(G_n) \setminus X_n\}. \quad (1)$$

Then by assumption (†13), v will be t_m , unless t_m was already in X_n anyway. In any case, since $|X_n| = |Y_n| = n$, it follows from (†11) that $i \leq n$, so by (†12), v does not lie in our leaf sets $R_n \cup B_n$, i.e.

$$v \notin R_n \cup B_n. \quad (2)$$

In the next sections, we will demonstrate how to obtain graphs $G_{n+1} \supset G_n$, $H_{n+1} \supset H_n$ and F_{n+1} with $X_{n+1} = X_n \cup \{v\}$ and $Y_{n+1} = Y_n \cup \{\varphi_{n+1}(v)\}$ satisfying (†1)–(†10) and (†13)–(†15).

After we have completed this step, since $|\mathbb{N} \setminus J_n| = \infty$, it is clear that we can extend our enumerations of G_n and H_n to enumerations of G_{n+1} and H_{n+1} as required, making sure to first list some new elements that do not lie in $R_{n+1} \cup B_{n+1}$. This takes care of (†11) and (†12) and completes the step $n \mapsto n+1$.

5.6. The inductive step: construction. We will construct the graphs G_{n+1} and H_{n+1} in three steps. First, in §5.6.1 we construct graphs $G'_{n+1} \supset G_n$ and $H'_{n+1} \supset H_n$ such that there is a vertex $\phi_{n+1}(v) \in H'_{n+1}$ with $G'_{n+1} - v \cong H'_{n+1} - \phi_{n+1}(v)$. This first step essentially follows the argument from [4, §4.6]. We will also construct a graph F_{n+1} via a parallel process.

Secondly, in §5.6.2 we will show that there are well-behaved maps from the coloured leaves of G'_{n+1} and H'_{n+1} to $F_{n+1} \square \mathbb{N}$, such that analogues of (†14) and (†15) hold for G'_{n+1} , H'_{n+1} and F_{n+1} , giving us control over the corresponding gluing sum.

Lastly, in §5.6.3, we do the actual gluing process and define all objects needed for step $n+1$ of our inductive construction.

5.6.1. Building the auxiliary graphs. Given the two graphs G_n and H_n , we extend each of them through their roots as indicated in Figure 7 to graphs \tilde{G}_n and \tilde{H}_n respectively.

Since v is not the root of G_n , there is a unique component of $G_n - v$ containing the root, which we call $G_n(r)$. Let $G_n(v)$ be the induced subgraph of G_n on the remaining vertices, including v . We remark that if v is not a cutvertex of G_n , then $G_n(v)$ is just a single vertex v . Since $\sigma_0(G_n) = k_n$ by (†5) and $\deg(v) \leq 5$ by (†2), it follows from an iterative application of Lemma 5.3 that $\Sigma(G_n(r))$ and $\Sigma(G_n(v))$ are finite. Let $k = \tilde{k}_n = \max\{\sigma_0(G_n), \sigma_0(G_n(r)), \sigma_0(G_n(v)), \sigma_0(H_n)\} + 1$.

To obtain \tilde{G}_n , we extend G_n through its root $r(G_n) \in R_n$ by a path

$$r(G_n) = u_0, u_1, \dots, u_{p-1}, u_p = r(\hat{H}_n)$$

of length $p = 4(\tilde{k}_n + 1) + 1$, where at its last vertex u_p we glue a rooted copy \hat{H}_n of H_n (via an isomorphism $\hat{z} \leftrightarrow z$), identifying u_p with the root of \hat{H}_n .

Next, we add two additional leaves at u_0 and u_p , so that $\deg(r(G_n)) = 3 = \deg(r(\hat{H}_n))$. Further, we add a leaf $r(G'_{n+1})$ at u_{2k+2} , which will be our new root for the next tree G'_{n+1} ; and another leaf g at u_{2k+3} . This completes the construction of \tilde{G}_n .

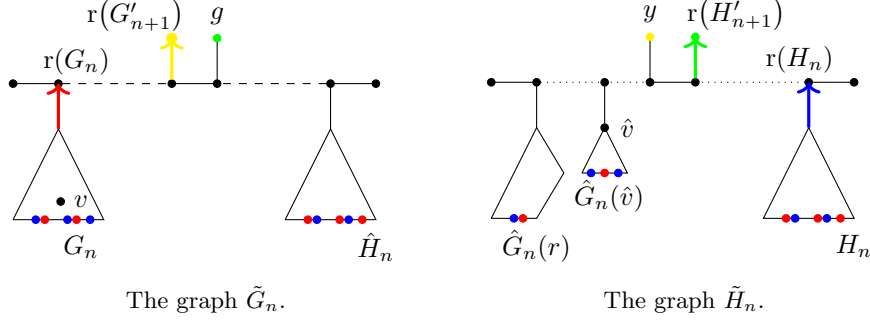


FIGURE 7. All dotted lines are mii-paths of length at least $k + 1 = \tilde{k}_n + 1$.

The construction of \tilde{H}_n is similar, but not entirely symmetric. For its construction, we extend H_n through its root $r(H_n) \in B_n$ by a path

$$r(H_n) = v_p, v_{p-1}, \dots, v_1, v_0 = r(\hat{G}_n(r))$$

of length p , where at its last vertex v_0 we glue a copy $\hat{G}_n(r)$ of $G_n(r)$, identifying v_0 with the root of $\hat{G}_n(r)$. Then, we take a copy $\hat{G}_n(\hat{v})$ of $G_n(v)$ and connect \hat{v} via an edge to v_{k+1} .

Finally, as before, we add two leaves at v_0 and v_p so that $\deg(r(\hat{G}_n(r))) = 3 = \deg(r(H_n))$. Next, we add a leaf $r(H'_{n+1})$ to v_{2k+3} , which will be our new root for the next tree H'_{n+1} ; and another leaf y to v_{2k+2} . This completes the construction of \tilde{H}_n .

By the induction assumption, certain leaves of G_n have been coloured with one of the two colours in $R_n \cup B_n$, and also some leaves of H_n have been coloured with one of the two colours in $R_n \cup B_n$. In the above construction, we colour leaves of \hat{H}_n , $\hat{G}_n(r)$ and $\hat{G}_n(\hat{v})$ accordingly:

$$\begin{aligned} \tilde{R}_n &= \left(R_n \cup \left\{ \hat{z} \in \hat{H}_n \cup \hat{G}_n(r) \cup \hat{G}_n(\hat{v}) : z \in R_n \right\} \right) \setminus \left\{ r(G_n), r(\hat{G}_n(r)) \right\}, \\ \tilde{B}_n &= \left(B_n \cup \left\{ \hat{z} \in \hat{H}_n \cup \hat{G}_n(r) \cup \hat{G}_n(\hat{v}) : z \in B_n \right\} \right) \setminus \left\{ r(H_n), r(\hat{H}_n) \right\}. \end{aligned} \quad (3)$$

Now put $M_n := \tilde{G}_n \cup \tilde{H}_n$ and consider the following promise structure $\mathcal{P} = (M_n, \vec{P}, \mathcal{L})$ on M_n , consisting of four promise edges $\vec{P} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4\}$ and corresponding leaf sets $\mathcal{L} = \{L_1, L_2, L_3, L_4\}$, as follows:

- \vec{p}_1 pointing in G_n towards $r(G_n)$, with $L_1 = \tilde{R}_n$,
- \vec{p}_2 pointing in H_n towards $r(H_n)$, with $L_2 = \tilde{B}_n$,
- \vec{p}_3 pointing in \tilde{G}_n towards $r(G'_{n+1})$, with $L_3 = \{r(G'_{n+1}), y\}$,
- \vec{p}_4 pointing in \tilde{H}_n towards $r(H'_{n+1})$, with $L_4 = \{r(H'_{n+1}), g\}$.

Note that our construction so far has been tailored to provide us with a \vec{P} -respecting isomorphism

$$h: \tilde{G}_n - v \rightarrow \tilde{H}_n - \hat{v}. \quad (5)$$

Consider the closure $\text{cl}(M_n)$ with respect to the above defined promise structure \mathcal{P} . Since $\text{cl}(M_n)$ is a leaf-extension of M_n , it has two connected components, just as M_n . We now define

$$\begin{aligned} G'_{n+1} &= \text{the component containing } G_n \text{ in } \text{cl}(M_n), \\ H'_{n+1} &= \text{the component containing } H_n \text{ in } \text{cl}(M_n). \end{aligned} \quad (6)$$

It follows that $\text{cl}(M_n) = G'_{n+1} \cup H'_{n+1}$. Further, since \vec{p}_3 and \vec{p}_4 are placeholder promises, $\text{cl}(M_n)$ carries a corresponding promise structure, cf. Def. 3.5. We define

$$R_{n+1} = \text{cl}(L_3) \quad \text{and} \quad B_{n+1} = \text{cl}(L_4). \quad (7)$$

Lastly, set

$$\begin{aligned} X_{n+1} &= X_n \cup \{v\}, \\ Y_{n+1} &= Y_n \cup \{\hat{v}\}, \\ \varphi_{n+1} &= \varphi_n \cup \{(v, \hat{v})\}, \\ k_{n+1} &= 2(\tilde{k}_n + 1). \end{aligned} \quad (8)$$

We now build F_{n+1} in a similar fashion to the above procedure. That is, we take two copies of F_n and join them pairwise through their roots as indicated in Figure 8 to form a graph \tilde{F}_n . We consider the graph $N_n = \tilde{F}_n \cup \hat{\tilde{F}}_n$, and take F_{n+1} to be one of the components of $\text{cl}(N_n)$ (unlike for $\text{cl}(M_n)$, both components of $\text{cl}(N_n)$ are isomorphic).

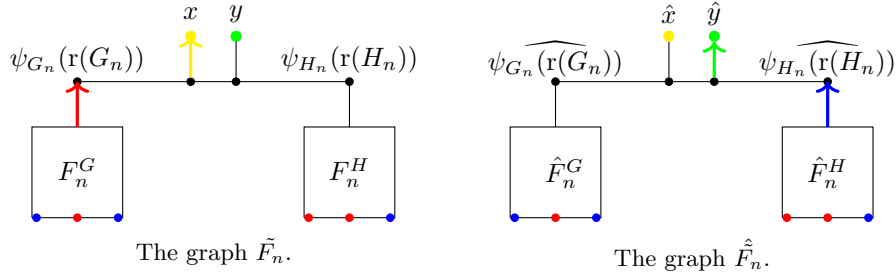


FIGURE 8. The graph $N_n = \tilde{F}_n \cup \hat{\tilde{F}}_n$.

More precisely we take two copies of F_n , which we will denote by F_n^G and F_n^H . We extend F_n^G through the image of the $r(G_n)$ under the bijection ψ_{G_n} by a path

$$\psi_{G_n}(r(G_n)) = u_0, u_1, u_2, u_3 = \psi_{H_n}(r(H_n))$$

of length three, where $\psi_{G_n}(r(G_n))$ is taken in F_n^G and $\psi_{H_n}(r(H_n))$ is taken in F_n^H . Further, we add a leaf x at u_1 , and another leaf y at u_2 . We will consider the graph $N_n = \tilde{F}_n \cup \hat{\tilde{F}}_n$ as in Figure 8 formed by taking two disjoint copies of \tilde{F}_n .

By the induction assumption, certain leaves of F_n have been coloured with one of the two colours in $R'_n \cup B'_n$. In the above construction, we colour leaves of $F_n^G, F_n^H, \hat{F}_n^G$ and \hat{F}_n^H accordingly:

$$\begin{aligned} \tilde{R}'_n &= \left\{ w \in F_n^G \cup F_n^H \cup \hat{F}_n^G \cup \hat{F}_n^H : w \in R'_n \right\} \setminus \left\{ \psi_{G_n}(r(G_n)), \psi_{G_n}(\widehat{r(G_n)}) \right\} \\ \tilde{B}'_n &= \left\{ w \in F_n^G \cup F_n^H \cup \hat{F}_n^G \cup \hat{F}_n^H : w \in B'_n \right\} \setminus \left\{ \psi_{H_n}(r(H_n)), \psi_{H_n}(\widehat{r(H_n)}) \right\}. \end{aligned} \quad (9)$$

Now consider the following promise structure $\mathcal{P}' = (N_n, \vec{P}', \mathcal{L}')$ on N_n , consisting of four promise edges $\vec{P}' = \{\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4\}$ and corresponding leaf sets $\mathcal{L}' = \{L'_1, L'_2, L'_3, L'_4\}$, as follows:

- \vec{r}_1 pointing in F_n^G towards $\psi_{G_n}(r(G_n))$, with $L'_1 = \tilde{R}'_n$,
 - \vec{r}_2 pointing in \hat{F}_n^H towards $\psi_{H_n}(r(H_n))$, with $L'_2 = \tilde{B}'_n$,
 - \vec{r}_3 pointing in \tilde{F}_n towards x , with $L'_3 = \{x, \hat{x}\}$,
 - \vec{r}_4 pointing in $\hat{\tilde{F}}_n$ towards \hat{y} , with $L'_4 = \{y, \hat{y}\}$.
- (10)

Consider the closure $\text{cl}(N_n)$ with respect to the promise structure \mathcal{P}' defined above. Since $\text{cl}(N_n)$ is a leaf-extension of N_n , it has two connected components, and we define F_{n+1} to be the component containing F_n^G in $\text{cl}(N_n)$. Since \vec{r}_3 and \vec{r}_4 are placeholder promises, $\text{cl}(N_n)$ carries a corresponding promise structure, cf. Def. 3.5. We define

$$R'_{n+1} = \text{cl}(L'_3) \cap F_{n+1} \quad \text{and} \quad B'_{n+1} = \text{cl}(L'_4) \cap F_{n+1}. \quad (11)$$

5.6.2. Extending maps. In order to glue $F_{n+1} \square \mathbb{N}$ onto G'_{n+1} and H'_{n+1} we will need to show that analogues of (†14) and (†15) hold for G'_{n+1} , H'_{n+1} and F_{n+1} . Our next lemma is essentially [4, Claim 4.13], and is an analogue of (†14). We briefly remind the reader of the details, as we need to know the nature of our extensions in our later claims.

Lemma 5.7. *There is a family of isomorphisms $\mathcal{H}'_{n+1} = \{h'_{n+1,x} : x \in X_{n+1}\}$ witnessing that $G'_{n+1} - x$ and $H'_{n+1} - \varphi_{n+1}(x)$ are isomorphic for all $x \in X_{n+1}$, such that $h'_{n+1,x}$ extends $h_{n,x}$ for all $x \in X_n$.*

Proof. The graphs G'_{n+1} and H'_{n+1} defined in (6) are obtained from \tilde{G}_n and \tilde{H}_n by attaching at every leaf in \tilde{R}_n a copy of the rooted graph $\text{cl}(M_n)(\vec{p}_1)$, and by attaching at every leaf in \tilde{B}_n a copy of the rooted graph $\text{cl}(M_n)(\vec{p}_2)$ by (cl.2).

From (5) we know that there is a \vec{P} -respecting isomorphism

$$h: \tilde{G}_n - v \rightarrow \tilde{H}_n - \varphi_{n+1}(v).$$

In other words, h maps promise leaves in $L_i \cap V(\tilde{G}_n)$ bijectively to the promise leaves in $L_i \cap V(\tilde{H}_n)$ for all $i = 1, 2, 3, 4$.

There is for each $\ell \in \tilde{R}_n \cup \tilde{B}_n \cup \{r(G_n), r(H_n)\}$ a $\text{cl}(\vec{P})$ -respecting isomorphism of rooted graphs

$$f_\ell: \text{cl}(M_n)(\vec{q}_\ell) \cong \text{cl}(M_n)(\vec{p}_i) \quad (12)$$

given by (cl.3) for $\ell \in (\tilde{R}_n \cup \tilde{B}_n)$, where i equals blue or red depending on whether $\ell \in \tilde{R}_n$ or \tilde{B}_n , and for the roots of G_n and H_n we have $\vec{q}_r = \vec{p}_i$ and the isomorphism is the identity. Hence, for each ℓ ,

$$f_{h(\ell)}^{-1} \circ f_\ell: \text{cl}(M_n)(\vec{q}_\ell) \cong \text{cl}(M_n)(\vec{q}_{h(\ell)})$$

is a $\text{cl}(\vec{P})$ -respecting isomorphism of rooted graphs. By combining the isomorphism h between $\tilde{G}_n - v$ and $\tilde{H}_n - \varphi_{n+1}(v)$ with these isomorphisms between each $\text{cl}(M_n)(\vec{q}_\ell)$ and $\text{cl}(M_n)(\vec{q}_{h(\ell)})$ we get a $\text{cl}(\vec{P})$ -respecting isomorphism

$$h'_{n+1,v}: G'_{n+1} - v \rightarrow H'_{n+1} - \varphi_{n+1}(v).$$

To extend the old isomorphisms $h_{n,x}$ (for $x \in X_n$), note that G'_{n+1} and H'_{n+1} are obtained from G_n and H_n by attaching at every leaf in R_n a copy of the rooted graph $\text{cl}(M_n)(\vec{p}_1)$, and similarly by attaching at every leaf in B_n a copy of the rooted graph $\text{cl}(M_n)(\vec{p}_2)$. By induction assumption ($\dagger 14$), for each $x \in X_n$ the isomorphism

$$h_{n,x}: G_n - x \rightarrow H_n - \varphi_n(x)$$

maps the red leaves of G_n bijectively to the red leaves of H_n , and the blue leaves of G_n bijectively to the blue leaves of H_n . Thus, by (12),

$$f_{h_{n,x}(\ell)}^{-1} \circ f_\ell: \text{cl}(M_n)(\vec{q}_\ell) \cong \text{cl}(M_n)(\vec{q}_{h_{n,x}(\ell)})$$

are $\text{cl}(\vec{P})$ -respecting isomorphisms of rooted graphs for all $\ell \in (R_n \cup B_n) \cap V(G_n)$. By combining the isomorphism $h_{n,x}$ between $G_n - x$ and $H_n - \varphi_n(x)$ with these isomorphisms between each $\text{cl}(M_n)(\vec{q}_\ell) = G'_{n+1}(\vec{q}_\ell)$ and $\text{cl}(M_n)(\vec{q}_{h_{n,x}(\ell)}) = H'_{n+1}(\vec{q}_{h_{n,x}(\ell)})$, we obtain a $\text{cl}(\vec{P})$ -respecting extension

$$h'_{n+1,x}: G'_{n+1} - x \rightarrow H'_{n+1} - \varphi_n(x). \quad \square$$

Our next claim should be seen as an approximation to property ($\dagger 15$). Recall that $\text{cl}(N_n)$ has two components $F_{n+1} \cong \hat{F}_{n+1}$.

Lemma 5.8. *There are colour-preserving bijections*

$$\psi_{G'_{n+1}}: V(G'_{n+1}) \cap (R_{n+1} \cup B_{n+1}) \rightarrow R'_{n+1} \cup B'_{n+1},$$

$$\psi_{H'_{n+1}}: V(H'_{n+1}) \cap (R_{n+1} \cup B_{n+1}) \rightarrow \hat{R}'_{n+1} \cup \hat{B}'_{n+1},$$

and a family of isomorphisms

$$\hat{\Pi}_{n+1} = \left\{ \hat{\pi}_{n+1,x}: F_{n+1} \rightarrow \hat{F}_{n+1}: x \in X_{n+1} \right\}$$

such that for each $x \in X_{n+1}$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{L}(G'_{n+1}) & \xrightarrow{h'_{n+1,x} \upharpoonright \mathcal{L}(G'_{n+1})} & \mathcal{L}(H'_{n+1}) \\ \psi_{G'_{n+1}} \downarrow & & \downarrow \psi_{H'_{n+1}} \\ \mathcal{L}'(F_{n+1}) & \xrightarrow{\hat{\pi}_{n+1,x} \upharpoonright \mathcal{L}'(F_{n+1})} & \mathcal{L}'(\hat{F}_{n+1}) \end{array}$$

Proof. Defining $\psi_{G'_{n+1}}$ and $\psi_{H'_{n+1}}$. By construction, we can combine the maps ψ_{G_n} and ψ_{H_n} to obtain a natural colour-preserving bijection

$$\psi: \mathcal{L}(M_n) \rightarrow \mathcal{L}'(N_n),$$

which satisfies the assumptions of Lemma 3.7. Thus, by Corollary 3.8, there are bijections

$$\alpha^i: \mathcal{L}(\text{cl}(M_n)(\vec{p}_i)) \rightarrow \mathcal{L}'(\text{cl}(N_n)(\vec{r}_i))$$

which are colour-preserving with respect to the promise structures $\text{cl}(\mathcal{P})$ and $\text{cl}(\mathcal{P}')$ on $\text{cl}(M_n)$ and $\text{cl}(N_n)$, respectively.

We now claim that ψ extends to a colour-preserving bijection (w.r.t. $\text{cl}(\mathcal{P})$)

$$\text{cl}(\psi): \mathcal{L}(\text{cl}(M_n)) \rightarrow \mathcal{L}'(\text{cl}(N_n)).$$

Indeed, by (cl.3), for every $\ell \in \tilde{R}'_n \cup \tilde{B}'_n$, there is a \vec{P}' -respecting rooted isomorphism

$$g_\ell: \text{cl}(N_n)(\vec{q}_\ell) \rightarrow \text{cl}(N_n)(\vec{r}_i), \quad (13)$$

where i equals blue or red depending on whether $\ell \in \tilde{R}'_n$ or \tilde{B}'_n . As in the case of (12) we define the maps g_r with $\vec{q}_r = \vec{r}_i$ for the roots of F_n^G and \hat{F}_n^H respectively to be the identity. Together with the rooted isomorphisms f_ℓ from (12), it follows that for each $\ell \in \tilde{R}_n \cup \tilde{B}_n \cup \{\text{r}(G_n), \text{r}(H_n)\}$, the map

$$\psi_\ell = g_{\psi(\ell)}^{-1} \circ \alpha^i \circ f_\ell: \mathcal{L}(\text{cl}(M_n)(\vec{q}_\ell)) \rightarrow \mathcal{L}(\text{cl}(N_n)(\vec{q}_{\psi(\ell)}))$$

is a colour-preserving bijection. Now combine ψ with the individual ψ_ℓ to obtain $\text{cl}(\psi)$. We then put

$$\psi_{G'_{n+1}} = \text{cl}(\psi) \upharpoonright G'_{n+1} \quad \text{and} \quad \psi_{H'_{n+1}} = \text{cl}(\psi) \upharpoonright H'_{n+1}.$$

Defining isomorphisms $\hat{\Pi}_{n+1}$. To extend the old isomorphisms $\pi_{n,x}$, given by the induction assumption, note that by (cl.2), F_{n+1} is obtained from F_n by attaching at every leaf in R'_n a copy of the rooted graph $F_{n+1}(\vec{r}_1)$, and similarly by attaching at every leaf in B'_n a copy of the rooted graph $F_{n+1}(\vec{r}_2)$. For each $x \in X_n$ let us write $\hat{\pi}_{n,x}$ for the map sending each $z \in F_n^G$ to the copy of $\pi_{n,x}(z)$ in \hat{F}_n^H . By the induction assumption ($\dagger 15$), for each $x \in X_n$ the isomorphism

$$\hat{\pi}_{n,x}: F_n^G \rightarrow \hat{F}_n^H$$

preserves the colour of red and blue leaves. Thus, using the maps g_ℓ from (13), the mappings

$$g_{\hat{\pi}_{n,x}(\ell)}^{-1} \circ g_\ell: \text{cl}(N_n)(\vec{q}_\ell) \cong \text{cl}(N_n)(\vec{q}_{\hat{\pi}_{n,x}(\ell)})$$

are $\text{cl}(\vec{P}')$ -respecting isomorphisms of rooted graphs for all $\ell \in R'_n \cup B'_n$. By combining the isomorphism $\pi_{n,x}$ with these isomorphisms between each $F_{n+1}(\vec{q}_\ell)$ and $\hat{F}_{n+1}(\vec{q}_{\hat{\pi}_{n,x}(\ell)})$, we obtain a $\text{cl}(\vec{P}')$ -respecting extension

$$\hat{\pi}_{n+1,x}: F_{n+1} \rightarrow \hat{F}_{n+1}.$$

For the new isomorphism $\hat{\pi}_{n+1,v}: F_{n+1} \rightarrow \hat{F}_{n+1}$, we simply take the ‘identity’ map which extends the map sending each $z \in \tilde{F}_n$ to $\hat{z} \in \hat{F}_n$.

The diagram commutes. To see that the new diagram above commutes, for each $x \in X_n$ it suffices to check that for all $\ell \in (R_n \cup B_n) \cap V(G_n)$ we have

$$\hat{\pi}_{n+1,x} \circ \psi_{G'_{n+1}} \upharpoonright \mathcal{L}(G'_{n+1}(\vec{q}_\ell)) = \psi_{H'_{n+1}} \circ h'_{n+1,x} \upharpoonright \mathcal{L}(G'_{n+1}(\vec{q}_\ell)),$$

which by construction of $\text{cl}(\psi)$ above is equivalent to showing that

$$\hat{\pi}_{n+1,x} \circ \psi_\ell = \psi_{h_{n,x}(\ell)} \circ h'_{n+1,x}.$$

By definition of ψ_ℓ this holds if and only if

$$\hat{\pi}_{n+1,x} \circ g_{\psi(\ell)}^{-1} \circ \alpha^i \circ f_\ell = g_{\psi(h_{n,x}(\ell))}^{-1} \circ \alpha^i \circ f_{h_{n,x}(\ell)} \circ h'_{n+1,x}.$$

Now by construction of $\hat{\pi}_{n+1,x}$ and $h'_{n+1,x}$, we have

$$\hat{\pi}_{n+1,x} \circ g_{\psi(\ell)}^{-1} = g_{\hat{\pi}_{n,x}(\psi(\ell))}^{-1} \quad \text{and} \quad f_{h_{n,x}(\ell)} \circ h'_{n+1,x} = f_\ell.$$

Hence, the above is true if and only if

$$g_{\hat{\pi}_{n,x}(\psi(\ell))}^{-1} \circ \alpha^i \circ f_\ell = g_{\psi(h_{n,x}(\ell))}^{-1} \circ \alpha^i \circ f_\ell.$$

Finally, this last line holds since $\psi(\ell) = \psi_{G_n}(\ell)$ and $\psi(h_{n,x}(\ell)) = \psi_{H_n}(h_{n,x}(\ell))$ by definition of ψ , and because

$$\hat{\pi}_{n,x} \circ \psi_{G_n}(\ell) = \psi_{H_n} \circ h_{n,x}(\ell)$$

by the induction assumption.

For $\hat{\pi}_{n+1,v}$ we see that, as above, it will be sufficient to show that for all $\ell \in (\tilde{R}_n \cup \tilde{B}_n) \cap V(\tilde{G}_n)$ we have

$$\hat{\pi}_{n+1,v} \circ \psi_\ell = \psi_{h'_{n+1,v}(\ell)} \circ h'_{n+1,v},$$

which reduces as before to showing that,

$$g_{\hat{\pi}_{n+1,v}(\psi(\ell))}^{-1} \circ \alpha^i \circ f_\ell = g_{\psi(h'_{n+1,v}(\ell))}^{-1} \circ \alpha^i \circ f_\ell.$$

Recall that, $\hat{\pi}_{n+1,v}$ sends each v to \hat{v} and also, since $h'_{n+1,v} \upharpoonright \tilde{G}_n = h$, the image of every leaf $\ell \in (\tilde{R}_n \cup \tilde{B}_n) \cap V(\tilde{G}_n)$ is simply $\hat{l} \in \hat{G}_n(v) \cup \hat{G}_n(r)$. Hence we wish to show that

$$g_{(\psi(\hat{\ell}))}^{-1} \circ \alpha^i \circ f_\ell = g_{\psi(\hat{l})}^{-1} \circ \alpha^i \circ f_\ell,$$

that is,

$$(\psi(\hat{\ell})) = \psi(\hat{l}),$$

which follows from the construction of ψ . \square

5.6.3. Gluing the graphs together. Let us take the cartesian product of F_{n+1} with a ray, which we simply denote by $F_{n+1} \square \mathbb{N}$. If we identify F_{n+1} with the subgraph $F_{n+1} \square \{0\}$, then we can interpret both $\psi_{G'_{n+1}}$ and $\psi_{H'_{n+1}}$ as maps from $\mathcal{L}(G'_{n+1})$ and $\mathcal{L}(H'_{n+1})$ to a set of vertices in $F_{n+1} \square \mathbb{N}$, under the natural isomorphism between \hat{F}_{n+1} and F_{n+1} .

Instead of using the function $\psi_{G'_{n+1}}$ directly for our gluing operation, we identify, for every leaf l in $\mathcal{L}(G'_{n+1})$ the *unique neighbour* of l with $\psi_{G'_{n+1}}(l)$. Formally, define a bijection $\chi_{G_{n+1}}$ between the neighbours of $\mathcal{L}(G'_{n+1})$ and $\mathcal{L}'(F_{n+1})$ via

$$\chi_{G_{n+1}} = \left\{ (z_1, z_2) : \exists l \in \mathcal{L}(G'_{n+1}) \text{ s.t. } z_1 \in N(l) \text{ and } \psi_{G'_{n+1}}(l) = z_2 \right\}, \quad (14)$$

and similarly

$$\chi_{H_{n+1}} = \left\{ (z_1, z_2) : \exists l \in \mathcal{L}(H'_{n+1}) \text{ s.t. } z_1 \in N(l) \text{ and } \psi_{H'_{n+1}}(l) = z_2 \right\}. \quad (15)$$

Since two promise leaves in G'_{n+1} or H'_{n+1} are never adjacent to the same vertex, $\chi_{G_{n+1}}$ and $\chi_{H_{n+1}}$ are indeed bijections. Moreover, since all promise leaves were proper, the vertices in the domain of $\chi_{G_{n+1}}$ and $\chi_{H_{n+1}}$ have degree at least 3. Using our notion of gluing-sum (see Def. 4.1), we now define

$$G_{n+1} := G'_{n+1} \oplus_{\chi_{G_{n+1}}} (F_{n+1} \square \mathbb{N}) \text{ and } H_{n+1} := H'_{n+1} \oplus_{\chi_{H_{n+1}}} (F_{n+1} \square \mathbb{N}). \quad (16)$$

We consider R_{n+1} , B_{n+1} , X_{n+1} and Y_{n+1} as subsets of G_{n+1} and H_{n+1} in the natural way. Then $\psi_{G_{n+1}}$ and $\psi_{H_{n+1}}$ can be taken to be the maps $\psi_{G'_{n+1}}$ and $\psi_{H'_{n+1}}$, again identifying \hat{F}_{n+1} with F_{n+1} in the natural way. We also take the roots of G_{n+1} and H_{n+1} to be the roots of G'_{n+1} and H'_{n+1} respectively

This completes the construction of graphs G_{n+1} , H_{n+1} , and F_{n+1} , the coloured leaf sets R_{n+1} , B_{n+1} , R'_{n+1} , and B'_{n+1} , the bijections $\psi_{G_{n+1}}$ and $\psi_{H_{n+1}}$, as well as $\varphi_{n+1}: X_{n+1} \rightarrow Y_{n+1}$, and $k_{n+1} = 2(\tilde{k}_n + 1)$. In the next section, we show the

existence of families of isomorphisms \mathcal{H}_{n+1} and Π_{n+1} , and verify that $(\dagger 1)$ – $(\dagger 15)$ are indeed satisfied for the $(n+1)$ th instance.

5.7. The inductive step: verification.

Lemma 5.9. *We have $G_n \subset G_{n+1}$, $H_n \subset H_{n+1}$, $\Delta(G_{n+1}), \Delta(H_{n+1}) \leq 5$, $\Delta(F_{n+1}) \leq 3$, and the roots of G_{n+1} and H_{n+1} are in R_{n+1} and B_{n+1} respectively.*

Proof. We note that $G_n \subset G'_{n+1}$ by construction. Hence, it follows that

$$G_n \subset G'_{n+1} \subset G'_{n+1} \oplus_{\chi_{G_{n+1}}} (F_{n+1} \square \mathbb{N}) = G_{n+1},$$

and similarly for H_n . Since we glued together degree 3 and degree 2 vertices, and $\Delta(G_n), \Delta(H_n) \leq 5$ and $\Delta(F_n) \leq 3$, it is clear that the same bounds hold for $n+1$. Finally, since the root of \tilde{G}_n was a placeholder promise, and R_{n+1} was the corresponding set of promise leaves in $\text{cl}(\tilde{G}_n)$, it follows that the root of G'_{n+1} is in R_{n+1} , and hence so is the root of G_{n+1} . A similar argument shows that the root of H_{n+1} is in B_{n+1} . \square

Lemma 5.10. *We have $\sigma_0(G_{n+1}) = \sigma_0(H_{n+1}) = k_{n+1}$.*

Proof. By construction we have that $\sigma_0(\tilde{G}_n) = \sigma_0(\tilde{H}_n) = k_{n+1}$. Since G'_{n+1} and H'_{n+1} are realised as components of the promise closure of M_n , and this was a proper extension, it is a simple check that $\sigma_0(G'_{n+1}) = \sigma_0(H'_{n+1}) = k_{n+1}$. Also note that $F_{n+1} \square \mathbb{N}$ has no mii-paths of length bigger than two, since the vertices of degree two in $F_{n+1} \square \mathbb{N}$ are precisely those of the form $(\ell, 0)$ with ℓ a leaf of F_{n+1} .

Since $G'_{n+1} \oplus_{\chi_{G_{n+1}}} (F_{n+1} \square \mathbb{N})$ is formed by gluing a set of degree-two vertices of $F_{n+1} \square \mathbb{N}$ to a set of degree-three vertices in G'_{n+1} , it follows that $\sigma_0(G_{n+1}) = k_{n+1}$ as claimed. A similar argument shows that $\sigma_0(H_{n+1}) = k_{n+1}$. \square

Lemma 5.11. *The graphs G_{n+1} and H_{n+1} are spectrally distinguishable.*

Proof. Since in \tilde{G}_n we have that all long mii-paths except for those of length k_{n+1} are contained inside G_n or \tilde{H}_n , it follows from our induction assumption $(\dagger 5)$ that $\sigma_1(\tilde{G}_n) = k_n$. However, in \tilde{H}_n , we attached $\tilde{G}_n(\hat{v})$ to generate an mii-path of length $\tilde{k}_n + 1$ in \tilde{H}_n (see Fig. 7), implying that

$$\sigma_1(\tilde{H}_n) = \tilde{k}_n + 1 > k_n = \sigma_1(\tilde{G}_n).$$

As before, since the promise closures G'_{n+1} and H'_{n+1} are proper extensions of \tilde{G}_n and \tilde{H}_n , they are spectrally distinguishable. Lastly, since $F_{n+1} \square \mathbb{N}$ has no leaves and no mii-paths of length bigger than two, the same is true for G_{n+1} and H_{n+1} . \square

Lemma 5.12. *The graphs G_{n+1} and H_{n+1} have exactly one end, and $\Omega(G_{n+1} \cup H_{n+1}) \subset \overline{R_{n+1} \cup B_{n+1}}$.*

Proof. By the induction assumption $(\dagger 8)$, we know that $\Omega(G_n \cup H_n) \subset \overline{R_n \cup B_n}$.

Claim. *The set $R_{n+1} \cup B_{n+1}$ is dense for G'_{n+1} .*

Consider a finite $S \subset V(G'_{n+1})$. We have to show that any infinite component C of $G'_{n+1} - S$ has non-empty intersection with $R_{n+1} \cup B_{n+1}$.

Let us consider the global structure of G'_{n+1} as being roughly that of an infinite regular tree, as in Figure 2. Specifically, we imagine a copy of G_n at the top level, at the next level are the copies of G_n and H_n that come from a blue or red leaf in

the top level, at the next level are the copies attached to blue or red leaves from the previous level, and so on.

With this in mind, it is evident that either C contains an infinite component from some copy of $H_n - S$ or $G_n - S$, or C contains an infinite ray from this tree structure. In the first case, we have $|C \cap (R_n \cup B_n)| = \infty$ by induction assumption. Since any vertex from $R_n \cup B_n$ has a leaf from $R_{n+1} \cup B_{n+1}$ within distance $k_{n+1} + 1$ (cf. Figure 7), it follows that C also meets $R_{n+1} \cup B_{n+1}$ infinitely often. In the second case, the same conclusion follows, since between each level of our tree structure, there is a pair of leaves in $R_{n+1} \cup B_{n+1}$. This establishes the claim.

Claim. *The set $R_{n+1} \cup B_{n+1}$ is dense for H'_{n+1} .*

The proof of the second claim is entirely symmetric to the first claim.

To complete the proof of the lemma, observe that $F_{n+1} \sqcup \mathbb{N}$ is one-ended, and with $R_{n+1} \cup B_{n+1}$, also $\text{dom}(\chi_{G_{n+1}}) \cup \text{dom}(\chi_{H_{n+1}})$ is dense for $G'_{n+1} \cup H'_{n+1}$ by our claims. So by Corollary 4.4, the graphs G_{n+1} and H_{n+1} have exactly one end. Moreover, since $R_{n+1} \cup B_{n+1}$ meets both graphs infinitely, it follows immediately that it is dense for $G_{n+1} \cup H_{n+1}$. \square

Lemma 5.13. *The graph G_{n+1} is a proper mii-extension of infinite growth of G_n at $R_n \cup B_n$ to length $k_n + 1$, and $\text{Ball}_{G_{n+1}}(G_n, k_n + 1)$ does not meet $R_{n+1} \cup B_{n+1}$. Similarly, H_{n+1} is a proper mii-extension of infinite growth of H_n at $R_n \cup B_n$ to length $k_n + 1$, and $\text{Ball}_{H_{n+1}}(H_n, k_n + 1)$ does not meet $R_{n+1} \cup B_{n+1}$. Hence, $(\dagger 9)$ and $(\dagger 10)$ are satisfied at stage $n + 1$.*

Proof. We will just prove the statement for G_{n+1} , as the corresponding proof for H_{n+1} is analogous.

Since G'_{n+1} is an $\left((\tilde{R}_n \cup \tilde{B}_n) \cap V(\tilde{G}_n)\right)$ -extension of \tilde{G}_n , it follows that G'_{n+1} is an

$$\left(\left((\tilde{R}_n \cup \tilde{B}_n) \cap V(G_n)\right) \cup \mathfrak{r}(G_n)\right) = \left((R_n \cup B_n) \cap V(G_n)\right)\text{-extension of } G_n. \quad (17)$$

However, from the construction of the closure of a graph it is clear that that G'_{n+1} is also an L' -extension of the supergraph K of G_n formed by gluing a copy of $\tilde{G}_n(\vec{p}_1)$ to every leaf in $R_n \cap V(G_n)$ and a copy of $\tilde{H}_n(\vec{p}_2)$ to every leaf in $B_n \cap V(G_n)$, where L' is defined as the set of inherited promise leaves from the copies of $\tilde{G}_n(\vec{p}_1)$ and $\tilde{H}_n(\vec{p}_2)$.

However, we note that every promise leaf in $\tilde{G}_n(\vec{p}_1)$ and $\tilde{H}_n(\vec{p}_2)$ is at distance at least $\tilde{k}_n + 1$ from the respective root, and so $\text{Ball}_{G'_{n+1}}(G_n, \tilde{k}_n) = \text{Ball}_K(G_n, \tilde{k}_n)$. However, $\text{Ball}_K(G_n, \tilde{k}_n)$ can be seen immediately to be an mii-extension of G_n at $R_n \cup B_n$ to length \tilde{k}_n , and since $\tilde{k}_n \geq k_n + 1$ it follows that $\text{Ball}_{G'_{n+1}}(G_n, k_n + 1)$ is an mii-extension of G_n at $R_n \cup B_n$ to length $k_n + 1$ as claimed.

Finally, we note that $R_{n+1} \cup B_{n+1}$ is the set of promise leaves $\text{cl}(\mathcal{L}_n)$. By the same reasoning as before, $\text{Ball}_{G'_{n+1}}(G_n, k_n + 1)$ contains no promise leaf in $\text{cl}(\mathcal{L}_n)$, and so does not meet $R_{n+1} \cup B_{n+1}$ as claimed. Furthermore, it doesn't meet any neighbours of $R_{n+1} \cup B_{n+1}$.

Recall that G_{n+1} is formed by gluing a set of vertices in $(F_{n+1} \sqcup \mathbb{N})$ to neighbours of vertices in $R_{n+1} \cup B_{n+1}$. However, by the above claim, $\text{Ball}_{G'_{n+1}}(G_n, k_n + 1)$

does not meet any of the neighbours of $R_{n+1} \cup B_{n+1}$ and so $\text{Ball}_{G_{n+1}}(G_n, k_n + 1) = \text{Ball}_{G'_{n+1}}(G_n, k_n + 1)$, and the claim follows.

Finally, to see that G_{n+1} is a leaf extension of G_n of infinite growth, it suffices to observe that $G_{n+1} - G_n$ consists of one component only, which is a superset of the infinite graph $F_n \square \mathbb{N}$. \square

Lemma 5.14. *There is a family of isomorphisms*

$$\mathcal{H}_{n+1} = \{h_{n+1,x}: G_{n+1} - x \rightarrow H_{n+1} - \varphi_{n+1}(x) : x \in X_{n+1}\},$$

such that

- $h_{n+1,x} \upharpoonright (G_n - x) = h_{n,x}$ for all $x \in X_n$,
- the image of $R_{n+1} \cap V(G_{n+1})$ under $h_{n+1,x}$ is $R_{n+1} \cap V(H_{n+1})$,
- the image of $B_{n+1} \cap V(G_{n+1})$ under $h_{n+1,x}$ is $B_{n+1} \cap V(H_{n+1})$ for all $x \in X_{n+1}$.

Proof. Recall that Lemma 5.7 shows that there exists such a family of isomorphisms between G'_{n+1} and H'_{n+1} . Furthermore, we have that

$$G_{n+1} := G'_{n+1} \oplus_{\chi_{G_{n+1}}} (F_{n+1} \square \mathbb{N}) \text{ and } H_{n+1} := H'_{n+1} \oplus_{\chi_{H_{n+1}}} (F_{n+1} \square \mathbb{N}).$$

where it is easy to check that $\chi_{G_{n+1}}$ and $\chi_{H_{n+1}}$ satisfy the assumptions of Lemma 4.2, since the functions $\psi_{G'_{n+1}}$ and $\psi_{H'_{n+1}}$ do by Lemma 5.8.

More precisely, given $x \in X_{n+1}$ and $h'_{n+1,x}$, it follows from Lemma 5.8 that

$$\chi_{H_{n+1}} \circ h'_{n+1,x} \circ \chi_{G_{n+1}}$$

extends to an isomorphism $\pi_{n+1,x}$ of F_{n+1} . Hence, by Lemma 4.2, $h'_{n+1,x}$ extends to an isomorphism $h_{n+1,x}$ from $G_{n+1} - x$ to $H_{n+1} - y$. That this isomorphism satisfies the three properties claimed follows immediately from Lemma 5.7 and the fact that $h_{n+1,x} \upharpoonright (G_n - x) = h'_{n+1,x} \upharpoonright (G_n - x)$. \square

Lemma 5.15. *There exist bijections*

$$\psi_{G_{n+1}}: V(G_{n+1}) \cap (R_{n+1} \cup B_{n+1}) \rightarrow R'_{n+1} \cup B'_{n+1}$$

and

$$\psi_{H_{n+1}}: V(H_{n+1}) \cap (R_{n+1} \cup B_{n+1}) \rightarrow R'_{n+1} \cup B'_{n+1},$$

and a family of isomorphisms

$$\Pi_{n+1} = \{\pi_{n+1,x}: F_{n+1} \rightarrow F_{n+1} : x \in X_{n+1}\},$$

such that

- $\pi_{n+1,x} \upharpoonright R'_{n+1}$ is a permutation of R'_{n+1} for each x ,
- $\pi_{n+1,x} \upharpoonright B'_{n+1}$ is a permutation of B'_{n+1} for each x , and
- for each $x \in X_{n+1}$, the corresponding diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(G_{n+1}) & \xrightarrow{h_{n+1,x} \upharpoonright \mathcal{L}(G_{n+1})} & \mathcal{L}(H_{n+1}) \\ \psi_{G_{n+1}} \downarrow & & \downarrow \psi_{H_{n+1}} \\ \mathcal{L}(F_{n+1}) & \xrightarrow{\pi_{n+1,x} \upharpoonright \mathcal{L}(F_{n+1})} & \mathcal{L}(F_{n+1}) \end{array}$$

I.e. for every $\ell \in V(G_{n+1}) \cap (R_{n+1} \cup B_{n+1})$ we have $\pi_{n+1,x}(\psi_{G_{n+1}}(\ell)) = \psi_{H_{n+1}}(h_{n+1,x}(\ell))$.

Proof. Since $R_{n+1}, B_{n+1} \subset G'_{n+1} \cup H'_{n+1}$, and $h_{n+1,x}$ extends $h'_{n+1,x}$ for each $x \in X_{n+1}$, this follows immediately from Lemma 5.8 after identifying \hat{F}_{n+1} with F_{n+1} . \square

This completes our recursive construction, and hence the proof of Theorem 1.2 is complete.

6. A NON-RECONSTRUCTIBLE GRAPH WITH COUNTABLY MANY ENDS

In this section we will prove Theorem 1.3. Since the proof will follow almost exactly the same argument as the proof of Theorem 1.2, we will just indicate briefly here the parts which would need to be changed, and how the proof is structured.

The proof follows the same back and forth construction as in Section 5.2, however instead of starting with finite graphs G_0 and H_0 we will start with two infinite graphs, each containing one free end. For example we could start with the graphs in Figure 9.

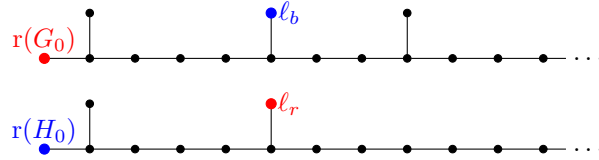


FIGURE 9. A possible choice for G_0 and H_0 , where the dots indicate a ray.

The induction hypotheses remain the same, with the exception of $(\dagger 7)$ and $(\dagger 8)$ which are replaced by

- $(\dagger 7')$ G_n and H_n have exactly one limit end and infinitely many free ends when $n \geq 1$, and
- $(\dagger 8')$ $\overline{R_n \cup B_n} \cap \Omega(G_n \cup H_n) = \Omega'(G_n \cup H_n)$.

The arguments of Section 5.5 will then go through mutatis mutandis: for the proof of the analogue of Lemma 5.12, use Corollary 4.5 instead of Corollary 4.4.

To show that the construction then yields the desired non-reconstructible pair of graphs with countably many ends, we have to check that $(\dagger 7')$ holds for the limit graphs G and H . It is clear that since $\overline{R_n \cup B_n} \cap \Omega(G_n \cup H_n) = \Omega'(G_n \cup H_n)$, every free end in a graph G_n or H_n remains free in the limit. Moreover, a similar argument to that in Section 5.3 shows that any pair of rays in G or H which were not in a free end in some G_n or H_n are equivalent in G or H , respectively.

However, since the end space of a locally finite connected graph is a compact metrizable space, and therefore has a countable dense subset, such a graph has at most countably many free ends, since they are isolated in $\Omega(G)$. Hence, both G and H have at most countably many free ends, and one limit end, and so both graphs have countably many ends.

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