Pursuit evasion on infinite graphs

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The cop-and-robber game is a game between two players, where one tries to catch the other by moving along the edges of a graph. It is well known that on a finite graph the cop has a winning strategy if and only if the graph is constructible and that finiteness is necessary for this result.

We propose the notion of weakly cop-win graphs, a winning criterion for infinite graphs which could lead to a generalisation. In fact, we generalise one half of the result, that is, we prove that every constructible graph is weakly cop-win. We also show that a similar notion studied by Chastand et al. (which they also dubbed weakly cop-win) is not sufficient to generalise the above result to infinite graphs.

In the locally finite case we characterise the constructible graphs as the graphs for which the cop has a so-called protective strategy and prove that the existence of such a strategy implies constructibility even for non-locally finite graphs.

1 Introduction

The game of cops and robbers was first studied by Nowakowski and Winkler [10] and Quilliot [14, 15]. Since its first appearance numerous variants and aspects of the game have been studied. The book [3] by Bonato and Nowakowski gives a fairly good overview.

The game is played on the vertex set of a graph between two players called the cop and the robber. Both players have perfect information. They alternately take turns, with the cop going first. In the first round their move is to select a starting vertex. In each consecutive round they can either move to a neighbour of the vertex they are at, or stay where they are.

The goal of the cop is to "catch" the robber, that is, to occupy the same vertex as the robber after finitely many steps. The robber wins if he can avoid being caught forever. Since one of the two events must happen, von Neumanns Theorem implies the existence of a winning strategy for one of the two players.

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While the focus has been mostly on finite graphs there have also been several publications treating the infinite case [1, 2, 6, 8, 11, 12, 13].

One of the reasons why infinite graphs have not received more attention is that many very basic results break down as soon as we leave the realm of finite graphs. The most striking example of this is probably that graphs which contain an isometric copy of an infinite path can never be cop-win. Even worse, the game cannot be won by any finite number of cops on such a graph. The robber's strategy would simply consist of starting "further out" along the path than any cop and then running away in a straight line. Clearly the cops can never catch up and hence the robber can avoid being captured forever.

But even if we remove this obvious obstruction, results fail to generalise. For example, Hahn et al. [8] showed that there are infinite chordal graphs of diameter 2 which are not cop-win. This contrasts the fact that finite chordal graphs are always cop-win.

In this paper we study an altered winning criterion which seems to be better adapted to infinite graphs. It coincides with the original winning criterion for finite graphs, but allows to generalise some results to infinite graphs. We only consider the most basic version of the game where one cop tries to catch one robber, but it is certainly possible that a similar approach works for more general variants of the game as well.

The following necessary and sufficient condition for the existence of a winning strategy for the cop on a finite graph was discovered independently by Nowakowski and Winkler [10] and Quilliot [14]. The main motivation for the present paper is that there does not seem to be a satisfactory generalisation of this result to infinite graphs.

Theorem 1 (Nowakowski and Winkler [10], Quilliot [14]). *A finite graph is cop-win if and only if it is constructible.*

Constructible here means that G can be constructed according to certain rules which will be explained in the next section. The characterisation does not remain valid for infinite graphs. One reason for this is, as mentioned, graphs with an isometric copy of an infinite path can never be cop-win.

While this problem was already addressed by Chastand et al. [6], their proposed solution turns out to be unsatisfactory. They introduced the notion of C-weakly cop-win graphs[1] where the cop wins the game if he can either catch the robber or chase him away. They proved that certain infinite constructible graphs are cop-win in this sense and asked whether the C-weakly cop-win graphs are exactly the constructible graphs.

In this paper we show that this is not the case if we use their exact definition. With the following slightly modified definition of weakly cop-win based on the same intuition we are able to prove that the cop has a winning strategy on every constructible graph: call a graph weakly cop win if there is a strategy for the cop which prevents the robber from visiting any vertex infinitely many times.

We also introduce protective strategies, which are a special kind of winning strategies and show that every graph which admits such a protective strategy is constructible.

[1]Of course, Chastand et al. call them weakly cop-win. The reason we call them C-weakly cop-win is that we would like to reserve the term weakly cop-win for the new winning criterion introduced in Section 6.

For locally finite graphs we can even show that being constructible is equivalent to the existence of a protective strategy.

Finally we investigate dismantable graphs (which in the finite case are exactly the constructible graphs) and give a sufficient condition for an infinite dismantable graph to be weakly cop-win.

The rest of this paper is structured as follows. After introducing some basic notions, we outline the rules of the game and give a proof of Theorem 1. We briefly discuss, why a similar characterisation is not possible for infinite cop-win graphs. In Section 5 we outline the approach of Chastand et al. and show that there is a locally finite constructible graph which is not C-weakly cop-win. We then proceed to introduce our modified definition of weakly cop-win graphs and prove the results mentioned above.

The main result of Section 7, and probably of the whole paper is Theorem 9 which states that every constructible graph is weakly cop-win. In Section 8 we introduce the protective strategies mentioned earlier and show that a locally finite graph is constructible if and only if it admits a protective strategy. Section 9 contains some results on dismantable graphs. We conclude the paper with some interesting open questions about weakly cop-win graphs.

2 Basic notions

Throughout this paper $G = (V, E)$ will denote a simple graph G with vertex set V and edge set E. Graph theoretical notions which are not explicitly defined will be taken from [7].

Since throughout most of the paper the vertex sets of the graphs in consideration are well ordered, we start by recalling some facts about well orders and ordinal numbers. All of those facts can be found in most standard text books on set theory, readers not familiar with ordinals and well orders see for example [16] for a more detailed introduction.

A *well order* of a set is a total order (i.e. any two elements are comparable) in which every subset has a minimal element.

One elementary fact about well orders is that in such an order there is no infinite descending chain, that is, every sequence $x_1 < x_2 < x_3 < \dots$ can only have finitely many elements.

It is also known that every well order is order isomorphic to some ordinal number where an *order isomorphism* is a bijective, order preserving function whose inverse is also order preserving. We denote by Ord the class of ordinal numbers. The ordinals themselves are well ordered with respect to being an initial piece of one another. This order is such that we can identify α (and thus also every well ordered set which is order isomorphic to α) with the set $\{\nu \in \text{Ord} \mid \nu < \alpha\}$ by means of an order isomorphism.

Every ordinal α has a successor which is obtained from α by adding a new element which is larger than all other elements. We call β a *successor ordinal* if it is the successor of some $\alpha \in \text{Ord}$ and write $\beta = \alpha + 1$ and $\alpha = \beta - 1$, respectively. If an ordinal is not a successor we call it a *limit ordinal*.

We identify the (up to order isomorphism) unique well order on an n-element set with the natural number $n \in \mathbb{N}$. The set $\mathbb{N}$ itself is also well ordered and can be identified with the smallest infinite ordinal number. If an order is order isomorphic to $\alpha \leq \mathbb{N}$ we call it a *natural order*.

Let $G = (V, E)$ be a graph and let V be well ordered. Then we can identify V with $\alpha = \{\nu \in \mathrm{Ord} \mid \nu < \alpha\}$ by means of an order isomorphism. In particular, the smallest vertex will always be denoted by 0. We denote by $G_{<\nu}$ the subgraph of G induced by all $\gamma < \nu$ and by $G_{\leq\nu}$ the subgraph of G induced by all $\gamma \leq \nu$. For example, $G_{<0}$ is the empty graph, $G_{<1}$ consists only of the vertex 0, and $G_{<V} = G$.

All graphs considered in this paper are undirected and reflexive, that is, there is a loop attached to each vertex. This is done mostly because it is convenient for defining the rules of the cops-and-robbers-game. Staying at a vertex simply amounts to taking a step along a loop in this vertex.

Let G, G' be graphs. A *homomorphism* from G to G' is an adjacency preserving map between the vertex sets. A *retraction* is a graph homomorphism from G onto a subgraph H of G whose restriction on H is the identity. We say that H is a *retract* of G if there is a retraction $\phi\colon G \to H$. Note that since all graphs in this paper are reflexive it is possible that adjacent vertices are mapped to the same vertex. In particular, every subgraph consisting of a single vertex with a loop attached is a retract of G.

3 Constructible graphs

Let $G = (V, E)$ be a graph and let $\nu, \mu \in V$. We say that $\mu \neq \nu$ *dominates* ν if μ is adjacent to ν and all of its neighbours.

A graph G is called *constructible* if there is a well order $<$ of its vertex set V, such that each vertex ν is dominated in $G_{\leq\nu}$. In other words, G can be constructed from a single vertex by recursively adding dominated vertices. It is not hard to see that every constructible graph is connected. An order $<$ with the aforementioned properties is called a *dominating order* or *constructing order*. A map δ which maps every vertex ν to a vertex which dominates ν in $G_{\leq\nu}$ is called a *domination map* associated to the dominating order $<$.

We wish to point out that our definition of dominating order is non-standard. Traditionally people have considered the inverse of this order, talking about dismantable rather than constructible graphs. A graph is *dismantable* if we can recursively remove dominated vertices until we end up with a single vertex, we call the order in which the vertices are removed a dismantling order. Clearly, dismantability and constructibility are equivalent conditions for finite graphs: reversing a dominating order always gives a dismantling order and vice versa.

This equivalence is not valid for infinite graphs. The reason is that both dominating and dismantling orders are required to be well orders, but reversing a well order in general does not give a well order. In this paper we use constructibility rather than dismantability because it seems to be better adapted to the inductive approach that we take. Nevertheless, in Section 9 we will consider dismantable graphs as well.

The following lemma about constructible graphs will be used later.

Lemma 2. *Let $G = (V, E)$ be a constructible graph with dominating order $<$ and an associated domination map δ. For every $\nu \in V$ there is $k \in \mathbb{N}$ such that $\delta^k(\nu) = 0$.*

Proof. The statement follows from the facts that $\delta(\nu) < \nu$ for every ν and that there are no infinite descending chains in a well order. Hence after finitely many iterations we must arrive at the minimal element 0. □

Next we would like to construct a retraction from G onto $G_{<\nu}$. Define a map ρ_ν by

$$\rho_\nu(\mu) = \delta^{k(\nu,\mu)}(\mu),$$

where

$$k(\nu, \mu) = \min\{k \in \mathbb{N} \mid \delta^k(\mu) < \nu\}.$$

Note that by the above lemma $k(\nu, \mu)$ is well defined and finite for every $\nu \geq 1$ because in this case $0 < \nu$.

Lemma 3. *Let $G = (V, E)$ be constructible and assume that V is ordered according to some constructing well order. The map ρ_ν is a retraction from G onto $G_{<\nu}$.*

Proof. First observe that $k(\nu, \mu) = 0$ if $\mu < \nu$. So in this case $\rho_\nu(\mu) = \mu$ and thus the restriction of ρ_ν to $G_{<\nu}$ is indeed the identity map.

It remains to show that ρ is a homomorphism, that is, adjacent vertices map to adjacent vertices or to the same vertex. This is done by transfinite induction. More precisely, for every $\gamma \in \mathrm{Ord}$ such that $\nu \leq \gamma \leq V$ we show that the restriction of ρ_ν to $G_{<\gamma}$ is a homomorphism from $G_{<\gamma}$ to $G_{<\nu}$.

For the base case $\gamma = \nu$ we get the identity map which clearly is a homomorphism. For the induction step assume that the statement was true for every $\gamma' < \gamma$. If γ is a limit ordinal, then each edge of $G_{<\gamma}$ appears in $G_{<\gamma'}$ for some $\gamma' < \gamma$ and hence is mapped to an edge of $G_{<\nu}$.

On the other hand, if γ is a successor ordinal, then $G_{<\gamma}$ is obtained from the predecessor $G_{<\gamma-1}$ by adding the vertex $\gamma - 1$. Furthermore there is a vertex $\mu = \delta(\gamma - 1)$ which dominates $\gamma - 1$ in $G_{<\gamma}$. By definition of ρ_ν it follows that $\rho_\nu(\gamma - 1) = \rho_\nu(\mu)$. Since every edge incident to μ is mapped to an edge of $G_{<\nu}$ and the neighbours of $\gamma - 1$ are a subset of the neighbours of μ we conclude that every edge incident to $\gamma - 1$ is also mapped to an edge of $G_{<\nu}$. Since these are the only edges of $G_{<\gamma}$ which were not present in $G_{<\gamma-1}$ this completes the induction step and thus also the proof of the lemma. □

Sometimes it is beneficial to have a natural order rather than an arbitrary well order. The following lemma shows that in the case of locally finite graphs we always can restrict ourselves to natural dominating orders.

Lemma 4. *A locally finite graph is constructible if and only if it admits a natural dominating order.*

Proof. The "if"-part is trivial, hence we only need to show that every constructible graph admits a natural dominating order.

For this purpose let $<$ be an arbitrary dominating order. We assign a number $n(\nu)$ to each vertex ν by transfinite recursion: $n(0) = 0$, and

$$n(\nu) = \max_{\substack{\mu \in N(\nu) \\ \mu < \nu}} n(\mu) + 1$$

for all other vertices, where $N(\nu)$ denotes (as usual) the neighbourhood of the vertex ν in G. Note that this maximum exists and is an integer since every vertex has only finitely many neighbours.

Order the vertices by their values of $n(\nu)$ where vertices with the same value are put in arbitrary order and denote the resulting order by $\prec$. We claim that $\prec$ is the desired dominating order.

It is easy to see that $\prec$ is order isomorphic to $\mathbb{N}$. Simply observe that $n(\nu)$ is at least as big as the distance between ν and 0. Hence for each value k there are only finitely many vertices with $n(\nu) = k$ and thus only finitely many vertices can appear before a given vertex in the order.

Finally observe that the neighbours of any vertex ν in $G_{\preceq\nu}$ are exactly the same as in $G_{\leq\nu}$ because every neighbour μ of ν with $\mu > \nu$ has $n(\mu) \geq n(\nu) + 1$. This implies that ν is dominated in $G_{\preceq\nu}$ by the same vertex as in $G_{\leq\nu}$. Since this is true for every vertex we have established that $\prec$ is dominating. □

4 The cop-and-robber game

The cop-and-robber game is a perfect information game played by 2 players—the cop and the robber. Following the notation of Bonato and Nowakowski [3], throughout this paper the cop will be female while the robber will be male. The players take turns in discrete time steps or *rounds* indexed by the non-negative integers. The cop plays in rounds with an even index while the robber plays in rounds with an odd index. The rules of the game are as follows.

First both players must choose their respective starting points, the cop in round 0 and the robber in round 1. In each subsequent round the players can move along one edge. It should be clear from the context what we mean when we say the cop is at (or occupies) a vertex $v \in V$ after round $n \in \mathbb{N}$.

The cop wins the game, if after some finite number of rounds both players occupy the same vertex, otherwise the robber wins.

Since one or the other must happen it follows from von Neumann's theorem that one of the two players has a winning strategy, where a *strategy* is a way to determine one's next move from the information each player has about the game, i.e. the current positions of the players as well as the moves in previous rounds that lead to this situation. Call a graph G *cop-win* if the cop has a winning strategy on G. Otherwise call it *robber-win.*

A *positional strategy* is a strategy where the cop's next move only depends on the current position and not on how this position was reached. As pointed out for example in [5], the cop has a positional winning strategy on every finite cop-win graph. We do not know whether this is still true in the infinite setting. Most of the strategies considered in this paper are positional, we will explicitly point out if a strategy uses more information.

It is an easy observation that if c dominates r, then the cop can win the game on her next turn whenever she occupies c and the robber occupies r. In the light of this it is not surprising that constructible graphs are cop-win. The following classical result which was already mentioned in Section 1 was discovered independently by Nowakowski and Winkler [10] and Quilliot [14]. We include a short proof, not only for the convenience of the reader, but also since we will reference some of the proof ideas later in this paper.

Theorem 1 (Nowakowski and Winkler [10], Quilliot [14])**.** *A finite graph is cop-win if and only if it is constructible.*

Proof. We use induction on the number of vertices. Clearly the statement is true for the graph on one vertex which is both constructible and cop-win. Now assume that the equivalence holds for every graph on at most $n-1$ vertices and let $G = (V, E)$ be a graph on n vertices.

First assume that G is cop-win. Then there must be vertices c and r occupied by the cop and the robber before the last move of the robber. It is easy to see that $G - r$ is cop-win. Simply note that the cop can play the exact same strategy as on G with the only difference that he uses c instead of r. By induction hypothesis $G-r$ is constructible.

We assume that the robber plays optimally, that is, he tries to avoid capture as long as possible. Then r is dominated by c because otherwise the robber could avoid being captured for at least one more round. By appending r to any constructing order of $G-r$ we hence obtain a constructing order of G.

Conversely assume that G is constructible. Let v be the last vertex in a dominating order of G. Then $G-v$ is constructible and hence cop-win. Furthermore there is a vertex $u \in V$ which dominates v.

Assume that the cop plays a winning strategy for $G - v$ on G, pretending that the robber is at u whenever he moves to v. Then after finitely many steps the cop either catches the robber, or she catches the robber's "shadow", meaning that the cop is at u while the robber is at v. In the latter case the cop will win the game in the next move, proving that G is cop-win. □

As mentioned earlier, traditionally the notion dismantability is used in place of constructibility in the above theorem.

5 Infinite graphs and the approach of Chastand et al.

Theorem 1 does not remain valid if we consider infinite graphs. The issue was first addressed by Chastand et al. [6]. They suggested the following change to the rules: the cop wins, if in order to avoid being caught, after some finite number of steps the robber has to move to a previously unvisited vertex on each move. The intuition behind this is that the robber can only avoid being caught by running away in a straight line from some point on.

Note that there is no a-priori bound on the number of steps after which the robber has to run away in a straight line. In fact it is easy to show that there cannot be such an a priori bound if the graph has infinite diameter. In this case, for any $n \in \mathbb{N}$ the robber could begin n steps away from the cop and only start running away once the cop approaches him. This will not happen before she has taken at least $n-1$ turns.

We call graphs for which the cop has a winning strategy with respect to this modified winning criterion *C-weakly cop-win.* Countable trees and locally finite chordal graphs are for example C-weakly cop-win. Furthermore, one can show the following result, where a *self contraction* is a homomorphism from G to itself. Recall that all graphs are assumed to be reflexive, hence a homomorphism maps adjacent vertices either to adjacent vertices or to the same vertex.

Theorem 5 (Chastand et al. [6]). *Let G be a constructible graph. Assume that G admits a dominating order $<$ such that there is a domination map associated to $<$ which is a self contraction of G. Then G is C-weakly cop-win.*

From the above theorem it follows that every Helly graph and every connected bridged graph is C-weakly cop-win. In the light of this result it is natural to ask if every constructible graph admits a dominating order as in the condition of Theorem 5 and whether or not the constructible graphs are exactly the C-weakly cop-win graphs (Questions 3 – 5 in [6]). This is however not the case as the following example shows.

Let G be a graph constructed as follows. Start with the 5-regular tree T_5. Then replace every vertex v of this tree by a copy the graph H shown in Figure 1 attaching the edges incident to v to the copies of the vertices $a_0, \dots a_4$. We would like to remark that the graph in Figure 1 has been discovered independently by Boyer et al. [4] and used to show that the distance between the cop and the robber in an optimal (with respect to the capture time) winning strategy of the cop is not necessarily monotonically decreasing. In fact, we also exploit that in order to catch the robber on this graph, the cop has to increase the distance to the robber giving him the opportunity to stay at the same vertex for another round.

Theorem 6. *The graph G defined above is constructible, but not C-weakly cop-win.*

Proof. To see that G is constructible pick an arbitrary root r of T_5. Clearly, the order in which vertices of T_5 are visited by a breadth first search starting at r is a dominating order.

Assume without loss of generality that all copies of H are inserted in a way that a_0 is the vertex closest to the copy of H that was used to replace the root. Observe that $a_0, b_4, c, b_0, \dots, b_3, a_1, \dots, a_4$ is a dominating order of H.

Put the copies of H in the same order as the corresponding vertices of T_5 appear in the breadth first search and order the vertices of each copy of H as above. Every copy a of a_0 is dominated in $G_{\leq a}$ by its only neighbour outside of the corresponding copy of H. On the other hand, if v is a copy of some other vertex of H, then the only neighbours of v in $G_{\leq v}$ lie in the corresponding copy of H. Since the order of the vertices of each copy of H is dominating, there must be a vertex which dominates v in $G_{\leq v}$.

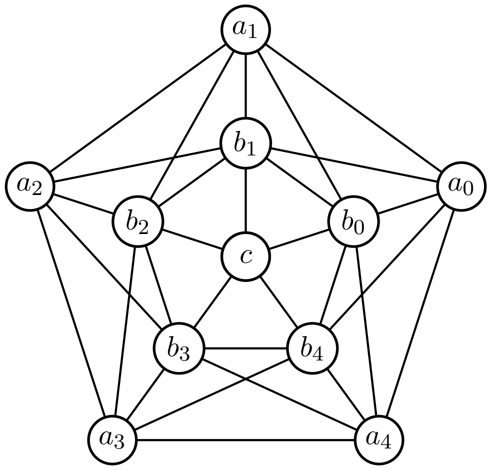

Figure 1: The building block H of the counterexample.

Hence it only remains to show that G is not C-weakly cop-win. To see this, first notice that the robber can run away forever without ever visiting a copy of b_i or c. So let us assume that the robber plays a strategy in which he only visits copies of the vertices a_i.

If after some finite number of steps the cop only uses vertices of type a_i and b_i, then the robber can avoid being caught forever while staying in one copy of H. The only thing he has to do is move around the outer cycle to stay as far away from the cop as possible. More precisely, when the cop is at a_i or b_i the robber moves to $a_{i+2 \mod 5}$ and when the cop leaves the copy of H in which the robber is then the robber stays where he is until the cop returns.

Thus the cop cannot win, unless she visits infinitely many copies of the center c. But whenever she does this, she is at least two steps away from the robber whose strategy makes him only visit copies of the vertices a_i. Clearly, each time this happens he has one step where he can choose to stay where he is or even backtrack before being forced to run to a previously unvisited vertex again. □

The last part of the above proof essentially uses the fact that (unless starting at the central vertex) in order to win on H, the cop must increase the distance between herself and the robber, c.f. Boyer et al. [4].

6 Weakly cop-win graphs

In the previous section we have seen that it is not possible to generalise Theorem 1 to infinite graphs using the notions of Chastand et al. In this section we suggest another definition of weakly cop-win graphs for which it is possible to generalise at least one of the two implications of the theorem. The intuition behind this definition is quite similar to the intuition behind the definition in the last section: The cop wins by either catching the robber, or by chasing him away.

We call a strategy of the cop *weakly winning* if it prevents the robber from visiting any vertex infinitely often. A graph G is called *weakly cop-win* if it admits a weakly winning strategy.

In particular, if the cop wins then in order to avoid being caught the robber must eventually leave every finite set of vertices. In all strategies we consider in this paper, the same is also true for the cop. We can think of this as "catching the robber at infinity" or "catching him a the boundary of G".

We would like to mention that there are also strategies for the cop, where, depending on the robbers strategy she stays inside some finite part of the graph as long as the robber behaves nicely, i.e., runs away reasonably fast and only commences pursuit if he fails to do so. In this case it may happen that the cop does not catch the robber at the boundary, because the robber may run away quickly enough for her to never pursue him. We do not know of any graph where all winning strategies for the cop are of this form and we do not expect there to be any.

The notion of weakly cop-win graphs can be used to generalise results about finite cop-win graphs to an infinite setting— sometimes even with the same proof as in the finite case. The following lemma is a nice example of such a result.

Lemma 7. *Retracts of weakly cop-win graphs are weakly cop-win.*

Proof. Let G be a weakly cop-win graph and let $\rho\colon G \to H$ be a retraction of G to H. Now imagine two games played in parallel on G and H where the robber is not allowed to leave the subgraph H.

The cop C_G on G plays the winning strategy while the cop C_H on H just shadows the moves of C_G with respect to the retraction ρ. That is, when C_G is at ν, then C_H will be at $\rho(\nu)$. It is clear that whenever C_G is inside the subgraph H, then the positions of G_G and C_H coincide. In particular, when C_G catches the robber, then so doesC_H because the robber never leaves H and hence capture must take place inside of H.

Now assume that the strategy of C_H is not winning. Then the robber can visit the same vertex infinitely many times without being caught by C_H. By the above observation this immediately implies that he is also not caught by C_G. So the robber's strategy is also winning in the graph G contradicting the assumption that the cop's strategy on G was winning. □

Note that the strategy we get from the above proof is not positional but indeed depends on the full history of the game as C_H has no possibility of deducing the position of C_G from her own position.

In the following theorem we restrict ourselves to locally finite graphs. This is done mainly to demonstrate how similar to the finite case the proof is. A more general version will be proved in the next section. The proof, however, is quite different.

Theorem 8. *If a locally finite graph G is constructible, then G is weakly cop-win.*

Proof. First note that by Lemma 4 we can assume that the dominating order on the vertex set is a natural order. In particular $G_{<\nu}$ is finite for every $\nu \in V$. Construct the strategy s_ν of the cop for each $G_{<\nu}$ exactly like in the proof of Theorem 1.

Note that for $\mu < \nu$ the strategies s_μ and s_ν coincide on $G_{<\mu}$. This observation is crucial because it allows us to define a limit strategy s on G by $s(c, r) = s_\nu(c, r)$ where $\nu > \max\{c, r\}$. Clearly, with this strategy the cop's position after her move is always smaller or equal to the robber's position because the same holds for each strategy s_ν.

Now assume that the robber can visit some vertex r infinitely many times. Since $G_{<r}$ is finite, this implies that there is some vertex $c \leq r$ such that it happens more than once that the cop is at c while the robber is at r.

But then the robber could repeat the steps between two occurrences of this situation arbitrarily. Since this leads to the same situations over and over again, neither the cop nor the robber will visit but finitely many vertices. Hence the robber would have a winning strategy in $G_{<\nu}$ where $\nu \in V$ is a common upper bound for all the vertices visited by the cop and the robber.

But this contradicts the fact that by Theorem 1 (or rather by its proof) the cop's strategy on $G_{<\nu}$ is winning. □

7 Constructible graphs, retractions, and another strategy

In this section we consider a different strategy s^* which is defined using the retraction maps $(\rho_\nu)_{\nu \in \mathrm{Ord}}$ from Lemma 3. In fact, the idea behind the strategy is not new: Isler et al. [9] describe essentially the same strategy for finite graphs.

Since ρ_ν is a retraction it is clear that if $uv \in E$ then $\rho_\nu(u)\rho_\nu(v) \in E$. For a fixed vertex v it follows from Lemma 2 that $\rho_\nu(v)$ only takes on finitely many different values $v, \delta(v), \delta^2(v), \ldots, \delta^k(v)$.

Now the strategy for the cop is to start at 0 (which coincides with $\rho_1(v)$ for every v). For each consecutive step define

$$s^*(c, r) = \delta^{k(c,r)}(r)$$

where

$$k(c, r) = \min\{k \in \mathbb{N} \mid \delta^k(r) \text{ is a neighbour of } c\}.$$

In other words the cop goes through the sequence $r, \delta(r), \delta^2(r), \ldots$ and moves to the first neighbour of c that she encounters.

To see that this is indeed a viable strategy recall that the cop starts at $0 = \rho_1(r)$. Now assume that the robber moves from r^{old} to r^{new}. If the cop is at $\rho_\nu(r^{\text{old}})$ for some $\nu \in \mathrm{Ord}$ then she can always move to $\rho_\nu(r^{\text{new}})$ because ρ_ν is a retraction. This implies that the above definition always yields a legal move as $\rho_\nu(r^{\text{new}}) = \delta^k(r^{\text{new}})$ for some $k \in \mathbb{N}$.

Theorem 9. *Every constructible graph is weakly cop-win.*

Proof. We will show that s^* is a winning strategy by bounding the number of visits of the robber to any particular vertex μ.

By the definition of the strategy, after the cop's move she is always at $\rho_\nu(r)$ where r is the position of the robber. Furthermore the cop can always follow the image of the robber under ρ_ν because ρ_ν is a retraction. This implies that the ν for which $c = \rho_\nu(r)$ is non-decreasing.

Now assume that in round $n-1$ the robber moved from r^{old} to r^{new} and that in round n the cop moved from c^{old} to c^{new}. There are integers k and l such that $c^{\text{old}} = \delta^k(r^{\text{old}})$ and $c^{\text{new}} = \delta^l(r^{\text{new}})$. Let $\xi = \delta^{k-1}(r^{\text{old}})$ and $\eta = \delta^{l-1}(r^{\text{new}})$.

If $\eta < \xi$ then $G_{<\xi}$ contains η but not ξ. Since $r^{\text{old}}r^{\text{new}}$ is an edge and ρ_ξ is a retraction this would imply that there is an edge connecting $c^{\text{old}} = \rho_\xi(r^{\text{old}})$ and $\eta = \rho_\xi(r^{\text{new}})$. But then the cop would have moved to η rather than to c^{new}. Furthermore, it is impossible that $\xi = \eta$ because in this case the cop would have moved to ξ rather than staying at $c^{\text{old}} = c^{\text{new}} = \delta(\xi)$.

Hence $\xi < \eta$ and thus $G_{<\eta}$ contains ξ but not η. This implies that $c^{\text{new}} = \rho_\eta(r^{\text{new}})$. Now recall that the ν for which $c = \rho_\nu(r)$ is non-decreasing. Hence if the robber visits r^{old} again, then the cop will be able to move to ξ and hence she will move to a vertex $c = \delta^{k'}(r^{\text{old}})$ for $k' < k$. In particular, the robber can visit the vertex r^{old} at most $k-1$ more times without being caught. □

Of course, the above proof also works for finite graphs. One may be tempted to think that this gives a different winning strategy for the cop than the proof of Theorem 1. While technically this is the case, the difference is not as big as one may think at the first glance. To prove our point, we define a second strategy for the cop on a constructible graph which is very similar to the strategy in the aforementioned proof.

It is defined by transfinite recursion on each graph $G_{<\nu}$. For $G_{<1}$ which only consists of the vertex 0 there is only one possible strategy s_1. Now assume that we have defined strategies s_γ on $G_{<\gamma}$ for each $\gamma < \nu$ such that for $\gamma' < \gamma < \nu$ the strategies s_γ and$s_{\gamma'}$ coincide on $G_{<\gamma'}$.

If ν is a limit ordinal then define $s_\nu(c,r) = s_\gamma(c,r)$ where $\gamma < \nu$ is chosen big enough that both c and r are contained in $G_{<\gamma}$. To see that this is well defined we can use the same line of argument as in the definition of the limit strategy in Theorem 8, that is, all strategies coincide on smaller retracts.

When ν is a successor ordinal then $G_{<\nu}$ is obtained from the predecessor $G_{<\nu-1}$ by adding the vertex $\nu-1$. There is a vertex $\mu = \delta(\nu-1)$ which dominates $\nu-1$ in $G_{<\nu}$. We now define

$$s_\beta(c,r) = \begin{cases} \nu - 1 & \text{if } r = \nu - 1 \text{ and } cr \in E, \\ s_{\beta-1}(c,\mu) & \text{if } r = \nu - 1 \text{ and } cr \notin E, \\ s_{\beta-1}(c,r) & \text{if } r \neq \nu - 1. \end{cases}$$

This defines a strategy on $G_{<\nu}$. We denote the strategy s_V on $G_{<V} = G$ obtained by the above construction by s. Note that just like the strategy s^* this strategy is not defined on all possible configurations (c,r). One implication of the following result is, however, that the strategy s is defined on all relevant configurations. The other implication is quite obvious: the strategy s^* is in fact very similar to the strategy used for the proof in the finite case.

Proposition 10. *The strategies s and s^* coincide.*

Proof. Consider the following setting: for every $\nu \in \mathrm{Ord}$ we have two cops C_ν and C^*_ν playing the strategies s and s^* respectively. Instead of playing against the robber, they play against the shadow R_ν of the robber: whenever the robber is at some vertex r then the shadow will be at the vertex $r^\nu = \rho_\nu(r)$. The game ends as soon as either of the cops has caught R_ν. Since in both strategies the cop's position after her move is always smaller or equal to the robber's position it is immediate that the game is really played on $G_{<\nu}$. This observation is crucial since it allows an inductive approach.

We claim that for any given trajectory of the robber, the corresponding trajectories of C_ν and C_ν^* coincide. This is sufficient to prove the proposition because for every position (c, r) which can occur in the game there is a finite number of steps leading to this position. Hence there is some $\nu \in \mathrm{Ord}$ such that the robber's trajectory until reaching this position is contained in $G_{<\nu}$ and thus coincides with the trajectory of R_ν. In other words, C_ν and C_ν^* have been playing against the real robber and hence

$$s(c, r) = s(c, r^\nu) = s^*(c, r^\nu) = s^*(c, r).$$

It only remains to prove the claim. We will use transfinite induction on ν. Furthermore, for each ν we will use induction on the number of steps taken by the robber.

Clearly, for $\nu = 1$ the claim is satisfied as $G_{<1}$ consists only of the vertex 0 and thus both of the cops win against the R_ν in the first step. In particular we do not need the induction on the robber's steps. For every $\nu > 1$ the first steps of C_ν and C_ν^* coincide as well because both strategies use 0 as a starting vertex.

For the induction step assume that the statement is true for every $\gamma < \nu$ and that the trajectories of C_ν and C_ν^* coincided until reaching some position (c, r). If there is some $\gamma < \nu$ such that $r^\gamma = r^\nu$, then by the induction hypothesis

$$s^*(c, r^\nu) = s^*(c, r^\gamma) = s(c, r^\gamma) = s(c, r^\nu)$$

and we are done. This is in particular the case if ν is a limit ordinal.

Now assume that this was not the case. Then ν is a successor ordinal and $r^\nu = \nu - 1$. If c is incident to $\nu - 1$ then both cops move straight to $\nu - 1$. Otherwise, in both strategies the cop will do the exact same thing as if the robber was at $\delta(\nu - 1)$. Since $\delta(\nu - 1) < \nu - 1$ we can use the induction hypothesis to show that the two strategies again must coincide. □

8 Protective strategies

In this section we would like to present a result towards the converse implication. To formulate this result we need a special kind of strategy for the cop, which will be called a *protective strategy*.

We say that the robber *robs a vertex ν at time n* if he is at ν after round n but does not get caught in round $n + 1$. Let s be a strategy of the cop. For every vertex ν define

$$t_r^s(\nu) = \sup\{t \in \mathbb{N} \mid \text{the robber robs } \nu \text{ at time } t\}$$

where the supremum is taken over t and over all possible strategies of the robber. Furthermore define

$$t_c^s(\nu) = \inf\{t \in \mathbb{N} \mid \text{the cop is at } \nu \text{ after round } t\}$$

where again the infimum is taken over t and over all strategies of the robber. Call a strategy protective, if

$$\forall \nu \in V : t_r^s(\nu) < t_c^s(\nu) < \infty.$$

The reason why we call this a protective strategy is that the cop protects an increasing proportion of the graph in the following sense. As soon as there is a possibility for the cop to visit a vertex according to the strategy, then the vertex cannot be robbed any more since an attempt to do so will lead to an immediate capture of the robber.

Note that a protective strategy is always winning, because $t_c^s(v)$ gives an a priori bound on how many times the robber can visit a certain vertex.

Theorem 11. *Let G be a graph on which there exists a protective strategy for the cop. Then G is constructible.*

Proof. We order the vertices by the values of $t_r^s(\nu)$. Vertices with equal values are put in an arbitrary order such that the resulting order $<$ is a well order. It is easy to see (e.g. using transfinite induction) that this is always possible.

We claim that $<$ is a dominating order. Let $\nu \in V$ and assume that the robber plays a strategy which allows him to rob ν at time $t_r^s(\nu)$.

Clearly, the cop must be at a neighbour μ of ν because otherwise the robber could stay at ν for another round which contradicts the definition of $t_r^s(\nu)$. Furthermore, if the robber moves to some vertex in $G_{\leq \nu}$ he will be caught immediately because $t_r^s(\eta) \leq t_r^s(\nu)$ for every $\eta < \nu$. This implies that μ is adjacent to every neighbour of ν in $G_{\leq \nu}$.

Finally we need to show that $\mu \in G_{\leq \nu}$. Since the cop is at μ at time $t_r^s(\nu)$ we have $t_c^s(\mu) \leq t_r^s(\nu)$ and because s is protective $t_r^s(\mu) < t_c^s(\mu)$ which implies that $\mu < \nu$ in the order. □

Theorem 12. *A locally finite graph G is constructible if and only if G admits a protective strategy.*

Proof. One of the two implications is a direct consequence of Theorem 11.

For the converse implication assume that G is constructible and let $<$ be a dominating order of the vertices. By Lemma 4 we can assume that $<$ is is order isomorphic to $\mathbb{N}$.

The strategy we would like to play is very similar to the strategy s^* in the proof of Theorem 9. The main difference is that we need to make sure that the cop does not move to a vertex too early (earlier than $t_r^s(\nu)$). For this purpose let δ be a domination map associated with the constructing order $<$ and let

$$\rho_\nu : G \to G_{<\nu}$$

be the retraction from Lemma 3. Now the strategy of c looks as follows:

$$s(c, r, 2n) = \rho_{n+1}(r).$$

The $2n$ in the above formula simply accounts for the fact that the cop only gets to move every second round.

If it is possible to play this strategy then it is easy to see that it is protective. Simply observe that c moves to a vertex n no earlier than round $2n$, thus $t_c^s(n) \geq 2n$. In fact it holds that $t_c^s(n) = 2n$ since $\rho_{n+1}(n) = n$ and thus a robber who just stays at vertex n will be caught after $2n$ steps. On the other hand, if the robber moves to n at some time $t > 2n - 1$, then he will be caught immediately because in this case $\rho_{\frac{t+1}{2}}(n) = n$. Hence he cannot rob n at any time $t > 2n - 1$.

It remains to show that the strategy defined above is indeed a viable strategy for the cop, that is, the cop can always move to $\rho_{n+1}(r)$. For this purpose it suffices to show that whenever μ and η are adjacent, there is also an edge connecting $\rho_n(\mu)$ and $\rho_{n+1}(\eta)$. Notice that there is an edge (possibly a loop) connecting $\rho_{n+1}(\mu)$ and $\rho_{n+1}(\eta)$ because ρ_{n+1} is a retraction by Lemma 3.

If $\rho_{n+1}(\mu) \neq n$, then $\rho_n(\mu) = \rho_{n+1}(\mu)$ and we are done. If $\rho_{n+1}(\mu) = n$, then $\rho_n(\mu)$ dominates $\rho_{n+1}(\mu)$ in $G_{\leq n}$. Thus every neighbour of $\rho_{n+1}(\mu)$ (in particular $\rho_{n+1}(\eta)$) is also a neighbour of $\rho_n(\mu)$.

This completes the proof that constructibility implies the existence of a time dependent protective strategy and thus also the proof of the theorem. □

We would like to point out that the protective strategy obtained for a dismantable graph in the above proof is in general not positional. It seems hard to get rid of the time dependence.

9 Dismantable graphs

We call a graph *dismantable* if there is a well order $<$ of the vertex set such that every vertex ν is dominated by some vertex μ in $G_{\geq \nu}$ where $G_{\geq \nu}$ is the subgraph of G induced by all vertices that are $\geq \nu$. The order $<$ is called a *dismantling order.* To this dismantling order we can associate a *domination map* which maps each vertex ν to some vertex which dominates it in $G_{\geq \nu}$.

Dismantability and constructibility are clearly equivalent for finite graphs, but they are quite different concepts for infinite graphs. The reason for this is that reversing a well order does not give a well order again unless the ordered set is finite.

It is relatively easy to see that there are infinite weakly cop-win graphs which are not dismantable. Simply consider any leafless tree. Since every vertex has at least two neighbours and those neighbours must be non-adjacent by acylicity of the tree there cannot be a dominated vertex in such a graph. Hence the tree is not dismantable. Conversely, it is easy to see that any tree is constructible and thus weakly cop-win.

One can also show that there are dismantable graphs which are not weakly cop-win. The following example is from [6]. Take an infinite path $a_0, a_1, a_2, \ldots$ and a finite path b_0, b_1, b_2 and connect b_0 and b_2 by an edge to every a_i. Denote the resulting graph by G. It is easy to see that $a_0, a_1, \ldots, b_0, b_1, b_2$ is a dismantling order.

To see that G is not weakly cop-win consider the map ρ defined by $\rho(a_i) = a_0$ for every $i \in \mathbb{N}$ and $\rho(b_i) = b_i$ for $0 \leq i \leq 2$. This map is a retraction from the graph G onto a 4-cycle. Since this cycle is not weakly cop-win we conclude by Lemma 7 that G cannot be weakly cop-win.

The above observations suggest that dismantability is likely not the right notion to use with our definition of weakly cop-win graphs. Nevertheless, in the remainder of this section we show that some of our proof techniques can be applied to dismantable graphs. In this respect, locally finite graphs seem to be particularly well behaved.

Let G be a dismantable graph with dismantling order $<$ and domination map δ. If possible we define a map $\rho_\nu \colon G \to G_{\geq \nu}$ by

$$\rho_\nu(\mu) = \delta^{k(\nu,\mu)}(\mu)$$

where

$$k(\nu, \mu) = \min\{k \in \mathbb{N} \mid \delta^k(\mu) \geq \nu\}.$$

Note that ρ_ν may not be defined for an arbitrary well order since $k(\nu, \mu)$ may well be infinite. Clearly whether or not ρ_ν is well defined does not only depend on the order isomorphism class of the dismantling order but also on the domination map. However, if the domination order is natural (that is, order isomorphic to a subset of $\mathbb{N}$) then the definition is always possible. An argument analogous to the proof of Lemma 4 shows that for locally finite graphs we can always find such an order. From now on we will restrict ourselves to natural dismantling orders although some of the results may be true in a more general context.

Just like in the constructible case we get that ρ_ν (if it is defined) is a retraction with some additional properties. We only prove this fact for graphs with a natural dismantling order.

Lemma 13. *Let G be a graph admitting a natural dismantling order. Then:*

- *The map ρ_ν it is a retraction.*
- *If μ and η are connected by an edge, then also $\rho_{\nu+1}(\mu)$ and $\rho_\nu(\eta)$ are connected by an edge.*

Proof. For the first part define the map $\delta_\nu \colon G_{\geq\nu} \to G_{\geq\nu+1}$ by $\delta_\nu(\nu) = \delta(\nu)$ and $\delta_\nu(\mu) = \mu$ for $\mu > \nu$. Clearly δ_ν is a retraction. Hence ρ_ν is a retraction because compositions of retractions are again retractions.

The second part is proved by induction. It is easy to check that the statement is true for $\nu = 1$. It is also clear that $\rho_\nu(\mu)$ is connected to $\rho_\nu(\eta)$ because ρ_ν is a retraction.

In the induction step we distinguish two cases. If $\rho_{\nu+1}(\mu) = \rho_\nu(\mu)$ then it is connected to $\rho_\nu(\eta)$. Otherwise $\rho_{\nu+1}(\mu)$ dominates $\nu = \rho_\nu(\mu)$ in $G_{\geq\nu}$. In particular $\rho_{\nu+1}(\mu)$ is connected by an edge to every neighbour of $\rho_\nu(\mu)$ and hence also to $\rho_\nu(\eta)$. □

Theorem 14. *Let G be a dismantable graph with a natural dismantling order $<$. Assume that ρ_ν is defined for every $\nu \in V$ and that for each pair ν, μ of vertices there is some γ such that $\rho_\gamma(\nu) = \rho_\gamma(\mu)$. Then G is weakly cop-win.*

Proof. Informally the cop's strategy can be summarised as follows: First she tries to get to a vertex $c = \rho_c(r)$ where r is the robber's position. Once she has achieved that, she follows a similar strategy as in Theorem 9 to catch the robber.

More formally let $k(c,r) = \min\{k \in \mathbb{N} \mid c\delta^k(r) \in E\}$ with the convention that the minimum over the empty set is ∞. The cop starts at 0. For each consecutive step we define a strategy by

$$s(c,r) = \begin{cases} \delta^{k(c,r)}(r) & \text{if } k(c,r) < \infty, \\ \delta(c) & \text{if } k(c,r) = \infty. \end{cases}$$

It remains to show that this strategy is weakly winning.

First we claim that if $k(c,r)$ is finite at some point, then the robber will be caught after finitely many rounds. To prove this claim observe that if $k(c,r)$ is finite, then after the cop's move $c = \rho_\gamma(r)$ for some suitable $\gamma \in \mathbb{N}$, where c and r are the positions of the cop and the robber respectively. By the second part of Lemma 13 we have $c = \rho_{\gamma'}(r)$ for some $\gamma' < \gamma$ after the next round. Continuing inductively we get $c = \rho_0(r) = r$ after at most γ rounds.

Thus we only need to show that $k(c,r)$ cannot remain infinite forever if the robber visits one vertex infinitely often. So assume that there was a vertex ν which the robber visits infinitely often. By assumption there is some finite γ such that $\rho_\gamma(\nu) = \rho_\gamma(0)$. Clearly, if $k(c,r) = \infty$ then the cop will arrive at $\rho_\gamma(0)$ after at most γ steps. The next time the robber visits ν the cop will be at $\delta^n(\rho_\gamma(0)) = \delta^m(\nu)$ for some $m, n \in \mathbb{N}$ which implies that $k(c,r) < \infty$. This completes the proof of the theorem. □

We would like to remark that a more careful analysis in the above proof shows that graphs satisfying the conditions of Theorem 14 are not only weakly cop-win, but even C-weakly cop-win. But since the requirements are so much stronger than in Theorem 9 it is still more than likely that constructibility is more useful than dismantability in the study of weakly cop-win graphs.

10 Outlook and open questions

While the present paper answers some of the questions posed in [6], the results raise several new questions about weakly cop-win graphs. The first and most obvious question is of course, whether Theorem 1 carries over to the infinite case with our notion of weakly cop-win graphs. Since one implication is already covered by Theorem 9 it only remains to answer the following question.

Question 15. *Is every weakly cop-win graph constructible?*

A first step towards the answer of this question could consist of treating locally finite graphs. One way to attack this question could be using Theorem 12 which in the locally finite case characterises the constructible graphs as the graphs for which there is a protective strategy.

Of course, one could also hope for a similar characterisation in the general case as a next step towards the above question.

Question 16. *Is there a similar characterisation to Theorem 12 for arbitrary (non-locally finite) graphs?*

Finally, one may also ask, under what circumstances a cop win strategy can be adapted into a protective strategy. In particular, a positive answer to the following question would immediately imply that the constructible graphs are exactly the weakly cop-win graphs.

Question 17. *Does every weakly cop-win graph admit a protective strategy?*

Questions 16 and 17 seem quite strong, so we expect the answers to be negative. Nevertheless it would be good to see an actual example.

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