On the cop number of toroidal graphs

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Abstract

We show that the cop number of toroidal graphs is at most 3. This resolves a conjecture by Schroeder from 2001 which is implicit in a question by Andreae from 1986.

1 Introduction

Cops and Robber is a pursuit-evasion game played on a graph between two players. Originally introduced independently by Nowakowski and Winkler [9], and Quilliot [10], this game and variants thereof have become a quickly growing research area within graph theory, see the book [5] for an extensive introduction to the topic.

The variant considered in this paper was first studied by Aigner and Fromme [1] and can be described as follows. Initially, the first player, called Cops, places k cops[1] on the vertices of a graph G. Then the second player, called Robber, places a robber on a vertex. Then the two players take turns. On Cops' turn each cop can either be moved to an adjacent vertex or left at the current position, on Robber's turn the robber can either be moved to an adjacent vertex or left where he is. Both players have perfect information, that is, they know the other player's moves and possible strategies. Cops wins the game if at some point one of the cops is at the same vertex as the robber, in this case we say that the robber is caught.

One of the most studied questions concerning this game is whether for some given k there is a winning strategy for Cops using k cops. The *cop number* of a graph G, usually denoted by $c(G)$ is the least k for which there is such a strategy. The most famous open problem in this context is Meyniel's conjecture, stating that the cop number of any graph on n vertices is at most $O(\sqrt{n})$. If true, this is asymptotically sharp since there are graph classes meeting this bound, however, not even an upper bound of the form $O(n^{1-\epsilon})$ is known, see [3] for an overview.

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[1] Throughout this note, we use Cops to refer to the player, and cops to refer to the playing pieces of that player on the graph. An analogous distinction is made between Robber and robber.

Bounds for the cop number have also been studied in certain graph classes, with an increased recent interest in graph classes defined by topological invariants, see for example the survey [4]. By a result by Andreae [2], for any fixed graph H there is a constant upper bound on the cop number of connected graphs with no H-minor. It follows that there is a constant upper bound on the cop number of connected graphs of genus g. In his paper, Andreae also poses the question of finding sharp bounds for the cop number of such graphs in terms of g.

So far, such a bound is only known for $g = 0$. Aigner and Fromme [1] showed that on any connected planar graph Cops has a winning strategy using 3 cops, and there are planar graphs (e.g. the dodecahedron) for which 3 cops are necessary. For toroidal graphs G, Quilliot [11] provided an upper bound of $c(G) \leq 5$, and Andreae [2] asked whether this could be improved to $c(G) \leq 3$. Schroeder [12] improved Quilliots bound to $c(G) \leq 4$, and explicitly stated the conjecture implicit in Andreae's question.

Conjecture 1.1 (Andreae, Schroeder)**.** *Let G be a finite toroidal graph, then $c(G) \leq 3$.*

In this short note we prove this conjecture. This is done by relating Cops and Robber on a graph G to a similar game with more powerful cops (which we call T-Cops and Robber) on a cover of G. We note that similar ideas have been used in [7], but without increasing the cops' power which is crucial for our proof to work.

As an application of our main result, we are able to make progress on the following conjecture of Schroeder [12].

Conjecture 1.2 (Schroeder)**.** *Let G be a finite graph of genus g, then $c(G) \leq g + 3$.*

The best known general bound is $c(G) \leq \lceil \frac{4}{3}g + 3 \rceil$, proved in [6], but so far the conjecture is only known to hold for $g \leq 2$. We give a simpler proof for the case $g = 2$, and prove the case $g = 3$.

While the above result confirms Conjecture 1.2 for $g \leq 3$, the bound is only known to be sharp for $g = 0$. Sharpness fails for $g = 1$ by our main result, thus raising the following question.

Question 1.3. *Is there any graph with genus $g > 0$ and cop number equal to $g + 3$?*

Maybe even more fundamentally, we do not know whether the bound in Conjecture 1.2 is asymptotically tight (Mohar [8] conjectured that it is not), which shows how little is known about the interplay between the genus and the cop number of a graph. In fact, to our best knowledge even the following question is still open.

Question 1.4. *What is the smallest g such that there is a graph with genus g and cop number 4?*

2 Preliminaries

Throughout this paper, let $G = (V, E)$ be a graph. All graphs considered are simple, undirected, and locally finite (i.e. every vertex only has finitely many neighbours).

An *embedding* of G on a surface S assigns to each vertex v a point p_v on S and to each edge $e = uv$ an arc a_e connecting p_u to p_v such that

- the points $(p_v)_{v\in V}$ are distinct,
- the arcs $(a_e)_{e\in E}$ are internally disjoint, and
- no point p_v lies in the interior of an arc a_e.

Clearly, given a set of points and arcs on a surface with the above properties we can find a graph with this embedding. A graph is called *planar*, if it has an embedding in the plane $\mathbb{R}^2$ and *toroidal*, if it has an embedding in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Let d denote the usual graph distance on V, that is, $d(u,v)$ is the length of a shortest path from u to v. For $v \in V$ and $r \in \mathbb{N}$ we define the *ball around v with radius r* by $B_v(r) = \{w \in V \mid d(v,w) \leq r\}$. The ball with radius 1 around v is called the *closed neighbourhood of* v and denoted by $N[v]$. A graph $\hat{G} = (\hat{V}, \hat{E})$ is a *cover* of G, if there is a surjective map $\phi : \hat{V} \to V$ such that ϕ is a bijection from $N[v]$ to $N[\phi(v)]$ for every $v \in \hat{V}$. The map ϕ is called a *covering map.* The *growth function* of G around v is the function $g \colon \mathbb{N} \to \mathbb{N}$ defined by $g(n) = |B_v(n)|$. We say that a graph has *polynomial growth*, if the growth function around some (or equivalently any) of its vertices is upper bounded by a polynomial.

The Cops and Robber game on G with k cops is a game played on G between two players, who are called Cops and Robber respectively. In the beginning of the game, Cops picks $(c_0^1, c_0^2, \dots, c_0^k) \in V^k$, then Robber picks $r_0 \in V$. In each subsequent turn n, Cops picks $c_n^i \in N[c_{n-1}^i]$, then Robber picks $r_n \in N[r_{n-1}]$. Cops wins the game, if $c_{n+1}^i = r_n$ or $c_n^i = r_n$ for some $n \in \mathbb{N}$ and some $1 \leq i \leq k$. Note that an optimally playing Robber can make sure that the latter option does not happen first, whence we could also insist on $c_n^i = r_{n-1}$ as a winning criterion. The *cop number* $c(G)$ is the least k such that Cops has a winning strategy.

Intuitively, we think of the c_n^i and r_n as the position of playing pieces on the graph, Cops' playing pieces are thought of as k cops, Robber's piece is thought of as a robber. Using this intuition, the winning criterion for Cops says that there some cop catches the robber by moving to the same vertex. We say that a subgraph H of G is *i-guarded at time n*, if $r_n \in H$ implies that $c_{n+1}^i = r_n$. Intuitively this means that Cops is using the i-th cop to make sure that the robber cannot move to H without being caught. Call a subgraph H *guarded*, if it is i-guarded for some $i \leq k$.

3 Main result

The following result is almost trivial and probably known, but we couldn't find a reference for it in the literature which is why we provide a proof sketch for the convenience of the reader.

Proposition 3.1. *Let G be a finite toroidal graph. Then there is an infinite planar cover of G with polynomial growth.*

Proof sketch. Let G be a toroidal graph and let $(p_v)_{v\in V}, (a_e)_{e\in E}$ be an embedding of G in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\pi \colon \mathbb{R}^2 \to \mathbb{T}^2$ be the usual projection map, i.e. $\pi(x) = x + \mathbb{Z}^2$.

For $v \in V$ define the set $P_v = \pi^{-1}(p_v)$, and for $e \in E$ let A_e be the set of connected components of $\pi^{-1}(a_e)$. In other words, P_v is the set of all points in $\mathbb{R}^2$ that project to the embedding p_v of v in $\mathbb{T}^2$, and A_e is a collection of arcs in $\mathbb{R}^2$ each of which projects to the embedding a_e of e in $\mathbb{T}^2$. It is readily verified that the set of points $P = \bigcup_{v \in V} P_v$ together with the set of arcs $A = \bigcup_{e \in E} A_e$ defines a drawing of a graph $\hat{G} = (\hat{V}, \hat{E})$ in the plane and that the projection π gives rise to a covering map by mapping $\hat{v}$ to v if $\pi(p_{\hat{v}}) = p_v$.

To show polynomial growth, note that in the embedding of $\hat{G}$ defined above, exactly $|V|$ vertices embed into any translate of $[0, 1)^2$. Since any two arcs in A_e can be mapped into each other by a translation, there is an absolute upper bound R on the Euclidian distance of the embeddings of two neighbours in $\hat{G}$. Consequently, the embeddings of all vertices in $B_v(r)$ are contained in some translate of $[-rR, rR + \epsilon)^2$ and thus $B_v(r)$ contains at most $(2rR + \epsilon)^2 \cdot |V|$ vertices. □

Given an equivalence relation T on V we can define the following variant of Cops and Robber, which we call T-Cops and Robber. The rules are the same as in the original game, except Cops is able to 'teleport cops to an equivalent position' before moving them, i.e. he can pick $\tilde{c}^i_{n-1} \, T \, c^i_{n-1}$ and choose $c^i_n \in N[\tilde{c}^i_{n-1}]$. The *$T$-cop number* $c^T(G)$ is the least k such that Cops has a winning strategy using k cops in T-Cops and Robber.

Assertion 3.2. For the remainder of this section, let $\hat{G} = (\hat{V}, \hat{E})$ be a cover of $G = (V, E)$ with covering map ϕ, and let T be the equivalence relation defined by $v \, T \, w \iff \phi(v) = \phi(w)$

The following lemma is very similar to [7, Lemma 1].

Lemma 3.3. *Under Assertion 3.2 we have* $c(G) \leq c^T(\hat{G})$.

Proof. We play two games in parallel, namely Cops and Robber on G with k cops and T-Cops and Robber on $\hat{G}$ with k cops. Let the positions in these games be $(c^i_n)_{i \leq k}, r_n$, and $(\hat{c}^i_n)_{i \leq k}, \hat{r}_n$ respectively.

Since ϕ is a covering map, for any play of Robber on G there is a unique way (up to choice of $\hat{r}_0$) for Robber on $\hat{G}$ to ensure that $\phi(\hat{r}_n) = r_n$. Assume that Robber on $\hat{G}$ plays such a strategy. Conversely, for any play of Cops on $\hat{G}$, there is a unique play for Cops on G such that $\phi(\hat{c}^i_n) = c^i_n$ for every i. Assume that Cops plays this strategy.

With the above setup it is obvious that if Cops wins the game on $\hat{G}$, then Cops wins the game on G in the same move or earlier. In particular, since Cops has a winning strategy on $\hat{G}$ for any $k \geq c^T(\hat{G})$ we conclude that the same is true for Cops on G, whence $c(G) \leq c^T(\hat{G})$. □

A weaker version of the next lemma can be found in [1]. The advantage of our version is that we can use the additional power of Cops in T-Cops and Robber to obtain a bound the distance between r_0 and r_j until the path P is guarded. This will be essential in the proof of our main result.

Lemma 3.4. *Assume Assertion 3.2, let $u, v \in \hat{V}$, let P be a shortest u-v-path, and let D be the length of P.*

*Let r_n, c_n^i be positions in T-*Cops and Robber *on $\hat{G}$ with k cops at time n such that $d(u, r_n) \leq D - |V|$ (in particular, G is finite), and let $i_0 \in \{1, \ldots, k\}$. Then there is a strategy for* Cops *such that for some $m > n$ the following hold:*

- *$d(u, r_j) \leq d(u, r_n) + |V|$ for $n \leq j \leq m$,*
- *P is i_0-guarded at all times $j \geq m$.*

Furthermore this strategy does not depend on how c_j^i evolve for $i \neq i_0$.

Proof. Without loss of generality take $i_0 = 1$ and $n = 0$. We give a strategy with the desired properties. Let x be the unique vertex on P satisfying $d(u, x) = d(u, r_0) + |V|$. Since ϕ is a covering map, it can be used to lift any path from $\phi(x)$ to $\phi(c_0^1)$ in G to a path from x to some $\tilde{c}_0^1 \, T \, c_0^1$ in $\hat{G}$. The distance between $\phi(x)$ and $\phi(c_0^1)$ in G is at most $|V|$, thus there is some $\tilde{c}_0^1 \, T \, c_0^1$ in $\hat{V}$ such that $d(x, \tilde{c}_0^1) \leq |V|$. Consequently, we can make sure that $c_{|V|}^1 = x$.

For $j > |V|$ we use the following strategy. If $d(u, r_j) < D$, then let r'_j be the unique vertex on P at the same distance from u as r_j, otherwise let $r'_j = v$. If $c_j^1 = r'_j$, then Cops chooses $c_{j+1}^1 = c_j^1$, otherwise c_{j+1}^1 is the unique neighbour of c_j^1 on P which lies closer to r'_j.

Clearly, r'_{j+1} is contained in the closed neighbourhood of r'_j in P, and it is easy to see that there is some m such that $c_{j+1}^1 = r'_j$ for every $j \geq m$. Take m minimal with this property. Trivially, $d(u, r_j) < d(u, r_0) + |V|$ for $j \leq |V|$. Since $r'_{|V|}$ lies closer to r_0 than $c_{|V|}^1 = x$, we know that the same is true for any j such that $|V| \leq j \leq m$, thus proving the first claimed property. The second property follows from the fact that if $r_j \in P$, then $r_j = r'_j = c_{j+1}^1$. Independence of this strategy from c_j^i for $i \neq 1$ is obvious, and thus the lemma is proved. □

The following lemma is obtained in the same way as Theorem 6 in [1], by starting the proof in situation (b), described on page 9 of [1].

Lemma 3.5. *Let G be a (possibly infinite) planar graph. Let P and Q be paths in G, and assume that in* Cops and Robber *with 3 cops, r_n lies in a finite component of $G - (P \cup Q)$. Further assume that* Cops *has a strategy such that both P and Q are guarded for every $j \geq n$. Then* Cops *has a winning strategy.*

Theorem 3.6. *Let $G = (V, E)$ be a finite toroidal graph, then $c(G) \leq 3$*

Proof. In Assertion 3.2, let $\hat{G}$ be a cover of G as in Proposition 3.1. By Lemma 3.3 it is enough to show that $c^T(\hat{G}) \leq 3$.

Assume that Cops and Robber have picked initial positions $(c_0^i)_{i \leq 3}$ and r_0 respectively. Choose D large enough that

$$\frac{D}{|V|} > \log(|B_{r_0}(D)|),$$

where log denotes the base 2 logarithm and $B_{r_0}(D)$ is the ball in $\hat{G}$. This is possible because $\hat{G}$ has polynomial growth and V is finite. Let $(v_i)_{1\leq i\leq l}$ be an enumeration of the vertices at distance D from r_0 in the cyclic order given by the embedding of $\hat{G}$ in $\mathbb{R}^2$. For convenience we define $v_0 = v_l$. Note that trivially $l < |B_{r_0}(D)|$.

Let T be a shortest path tree of $B_{r_0}(D)$ rooted at r_0, that is, the unique path in T connecting r_0 to v is a shortest r_0-v-path for every $v \in B_{r_0}(D)$. For $0 \leq i \leq l$, denote by P_i the path from r_0 to v_i in T.

For $a < b$, denote by $[a, b] = \{v_i \mid a < i < b\}$ and $[b, a] = \{v_i \mid i < a \text{ or } i > b\}$. Let H be the graph obtained from $B_{r_0}(D)$ removing the union of P_a and P_b. Note that H is disconnected, and in particular (since $\hat{G}$ is planar) that there is no path connecting $[a, b]$ to $[b, a]$ in H. We say that $v \in B_{r_0}(D)$ lies *between a and b* if it lies in the same component of H as some element of $[a, b]$. Observe that no vertex is between a and b and between b and a simultaneously, but there may be vertices in $B_{r_0}(D)$ which are neither between a and b nor between b and a. Clearly, any such vertex is contained in a finite component of $\hat{G} - (P_a \cup P_b)$, since any path connecting it to $\hat{G} - B_{r_0}(D)$ must cross either P_a or P_b.

We say that Robber is *trapped between a and b* at time j, if r_j lies between a and b, and P_a and P_b are guarded at time j. Note that in this case, Robber will remain trapped between a and b at time $j+1$ unless either $r_{j+1} \notin B_{r_0}(D)$, or $r_{j+1} \in P_a \cup P_b$ (in which case Cops wins the game), or Cops changes the strategy and stops guarding P_a or P_b.

We now inductively define for every integer $t \leq \frac{D}{|V|} - 1$ a value $n_t \in \mathbb{N}$, such that one of the following two statements holds.

(I) Cops has won the game before time n_t, or has a strategy to win starting from the position at time n_t.

(II) There are $a_t, b_t \in \mathbb{N}$ such that

- $1 \leq b_t - a_t \leq 1 + 2^{-(t+1)} \cdot l$, and
- Robber is trapped between a_t and b_t at time $n_t \leq j \leq n_{t+1}$.

Essentially, this is achieved using Lemma 3.4 to inductively guard P_y (for some appropriate y), thus trapping R, see Figure 1. We point out that the values n_t are not determined a priori, but depend on how the game evolves. In particular, different strategies of Robber may lead to different values for n_t on the same graph. Throughout the induction, we will also show that $d(r_0, r_{n_t}) \leq (t+1) \cdot |V|$, in order to make sure that $r_{n_t} \in B_{r_0}(D)$.

To start the inductive construction, let $y = \lfloor \frac{l}{2} \rfloor$. By Lemma 3.4 there is a strategy for Cops to make sure that P_0 is 1-guarded at all times $j > m$ and $d(r_0, r_m) \leq |V|$ for some appropriate m. Analogously there is a strategy to make sure that P_y is 2-guarded at all times $j > m'$ and $d(r_0, r_{m'}) \leq |V|$ for some appropriate m'. Since those two strategies don't interfere with each other, we have a strategy ensuring that both P_0 and P_y are guarded for $j \geq n_0$ and $d(r_0, r_{n_0}) \leq |V|$, where $n_0 := \max(m, m')$.

Let H be the graph obtained from $\hat{G}$ by removing P_0 and P_y. If $r_{n_0} \notin H$, then $r_{n_0} \in P_0 \cup P_y$ and Cops has won the game already. If r_{n_0} is in a finite component of

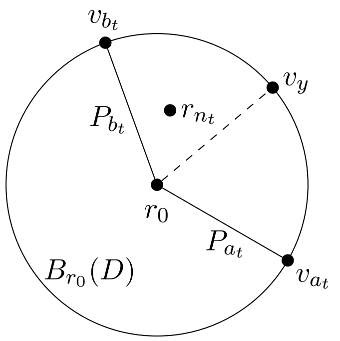

Figure 1: Situation at time j for $n_t \leq j \leq n_{t+1}$.

H, then COPS has a winning strategy by Lemma 3.5. In both of these cases (I) holds. Thus we can assume that r_{n_0} lies in an infinite component C of H. Since $r_{n_0} \in B_{r_0}(|V|)$, there must be a path in $C \cap B_{r_0}(|V|)$ connecting r_{n_0} to either $[0, y]$ or $[y, l]$ and thus at time n_0 ROBBER is trapped either between 0 and y or between y and l. In the first case choose $a_0 = 0$ and $b_0 = y$, in the second case choose $a_0 = y$ and $b_0 = l$. In both cases it is straightforward to check that (II) holds.

For the induction step assume that we have defined n_{t-1} as claimed. If (I) holds, then we can define $n_t = n_{t-1}$ and (I) still holds. So let us assume that (II) holds for n_{t-1}. Let $y = \lfloor \frac{a_{t-1}+b_{t-1}}{2} \rfloor$, let $i \in \{1, 2, 3\}$ be such that neither $P_{a_{t-1}}$ nor $P_{b_{t-1}}$ is i-guarded. Lemma 3.4 provides us with a strategy such that for an appropriate n_t we have that P_y is i-guarded at all times $j \geq n_t$, and $d(r_0, r_{n_t}) \leq d(r_0, r_{n_t}) + |V| \leq (t+1) \cdot |V|$.

Let H be the graph obtained from $\hat{G}$ by removing $P_{a_{t-1}}$, $P_{a_{t-1}}$ and P_y. As before, if $r_{n_t} \notin H$, then COPS has won the game and (I) holds. If r_{n_t} is contained in a finite component of H, then removing two of the three paths from G already leaves it in a finite component (because $\hat{G}$ is planar and $P_{a_{t-1}}$, $P_{a_{t-1}}$ and P_y pairwise don't cross in the embedding). Consequently, COPS has a winning strategy in this situation by Lemma 3.5. Finally assume that r_{n_t} is contained in an infinite component C of H. For $n_{t-1} \leq j \leq n_t$ the paths $P_{a_{t-1}}$ and $P_{b_{t-1}}$ are guarded at time j and $r_j \in B_{r_0}(D)$. Together with the assumption that ROBBER was trapped between a_{t-1} and b_{t-1} at time n_{t-1}, this implies that ROBBER is trapped between a_{t-1} and b_{t-1} at time n_t unless COPS has won the game before time n_t. Since P_y is now also guarded, the same argument as above gives that at time n_t ROBBER is either trapped between a_{t-1} and y, or between y and b_{t-1}. In the first case take $a_t = a_{t-1}$ and $b_t = y$, in the second case take $a_t = y$ and $b_t = b_{t-1}$. In both cases it is not hard to verify that (II) is satisfied.

To conclude the proof, we remark that the (II) can't possibly be satisfied for $t = \frac{D}{|V|} - 1$. Indeed, in this case

$$2^{-(t+1)} \cdot l < 2^{-\frac{D}{|V|}} \cdot |B_{r_0}(D)| = 2^{-\frac{D}{|V|} + \log(|B_{r_0}(D)|)} < 1.$$

Since $b_t - a_t$ is an integer, it follows that $b_t - a_t = 1$, and thus $[a_t, b_t] = \emptyset$. In particular r_{n_t} cannot lie between a_t and b_t. Hence there is some $t \leq \frac{D}{|V|} - 1$, such that (I) holds, thus COPS has a winning strategy. □

As mentioned in the introduction, Theorem 3.6 can be used to make progress on Conjecture 1.2. In particular, we have the following result.

Corollary 3.7. *If G is a finite graph of genus $g \leq 3$, then $c(G) \leq g + 3$.*

We remark that the cases $g \leq 2$ were previously known, see [1, 12], so our only real contribution here is the case $g = 3$. We still prove all cases for convenience. We say that a strategy of Cops *reduces the genus by r using s cops*, if it yields i-guarded subgraphs H_i of G for $1 \leq i \leq s$ such that the genus of the graph obtained from G by removing all H_i is at most $g - s$, where g is the genus of G. Using this notation, we have the following result.

Lemma 3.8. *Assume that we play* Cops and Robber *with $k \geq 4$ cops. Then*

1. Cops *has a strategy reducing the genus by 1 using 2 cops, and*
2. Cops *has a strategy reducing the genus either by 1 using 1 cop, or by 2 using 3 cops.*

Proof. The first part is implicit in [11], the second part is Proposition 3.2 in [12]. □

Proof of Corollary 3.7. For $g = 0$ and $g = 1$ this follows directly from Theorem 3.6 (note that any planar graph can be embedded in the torus). For $g = 2$ apply the first part of Lemma 3.8, then apply Theorem 3.6 to the resulting toroidal graph. For $g = 3$, first apply the second part of Lemma 3.8. If the strategy used 1 cop to reduce the genus by 1, then apply Corollary 3.7 for $g = 2$ to the remaining graph, otherwise apply Theorem 3.6. □

References

[1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Appl. Math.*, 8:1–11, 1984.

[2] T. Andreae. On a pursuit game played on graphs for which a minor is excluded. *J. Comb. Theory, Ser. B*, 41:37–47, 1986.

[3] W. Baird and A. Bonato. Meyniel's conjecture on the cop number: a survey. *J. Comb.*, 3(2):225–238, 2012.

[4] A. Bonato and B. Mohar. Topological directions in cops and robbers. 2018. Preprint.

[5] A. Bonato and R. J. Nowakowski. *The game of cops and robbers on graphs.*, volume 61. Providence, RI: American Mathematical Society (AMS), 2011.

[6] N. Bowler, J. Erde, F. Lehner, and M. Pitz. Bounding the cop number of a graph by its genus. 2019. Preprint.

[7] N. E. Clarke, S. Fiorini, G. Joret, and D. O. Theis. A note on the cops and robber game on graphs embedded in non-orientable surfaces. *Graphs Comb.*, 30(1):119–124, 2014.

[8] B. Mohar. Notes on cops and robber game on graphs. 2008. Unpublished note.

[9] R. Nowakowski and P. Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Math.*, 43:235–239, 1983.

[10] A. Quilliot. These de 3e cycle. *Université de Paris VI*, 1978.

[11] A. Quilliot. A short note about pursuit games played on a graph with a given genus. *J. Comb. Theory, Ser. B*, 38:89–92, 1985.

[12] B. S. W. Schroeder. The copnumber of a graph is bounded by $\lfloor \frac{3}{2} \operatorname{genus}(G) \rfloor + 3$. In *Categorical perspectives. Papers from the international conference held in honor of George E. Strecker on the occasion of his 60th birthday at Kent State University*, pages 243–263. Boston, MA: Birkhäuser, 2001.